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The MHV QCD Lagrangian

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Abstract: We perform a canonical change of the field variables of light-cone gauge massless QCD to obtain a lagrangian whose terms are proportional up to polarisation factors to MHV amplitudes and continued off shell by the CSW prescription. We solve for this transformation as a series expansion to all orders in the new fields, and use this to prove that the resulting vertices are indeed MHV vertices as claimed. We also demonstrate how this works explicitly for the vertices with: two quarks and two gluons, four quarks, and a particular helicity configuration of two quarks and three gluons. Finally, we generalise the construction to massive QCD.

Keywords: QCD, Gauge Symmetry.

Contents

1. Introduction

The calculation of scattering amplitudes for perturbative gauge theories has seen several advancements recently, both in terms of known analytic expressions and the computational techniques underpinning them. This is largely due to the fact that these theories appear to have a much simpler structure than the traditional Feynman graph computations would suggest. Parke and Taylor obtained very compact expressions for the MHV (and $\overline{\mathrm{MHV}}$) amplitudes [1–3]. Later, Cachazo, Svrček and Witten (CSW) discovered [4] the MHV rules:

inspired by twistor-string theory [5], they observed that tree-level amplitudes of arbitrary helicity configurations could be assembled from Parke-Taylor amplitudes, continued off shell by a particular prescription, and connected by scalar propagators. An important related development came in the form of the BCFW recurrence relations between S-matrix elements, discovered by Britto, Cachazo, Feng and Witten, described in [6] and proved in [7]. These can be viewed as an indirect proof of CSW's MHV rules (proved directly in [8]). In [9], Wu and Zhu extended the MHV rules to calculate the amplitudes with quarks and/or gluinos.

The MHV rules are tantalisingly suggestive of a field theory and have since seen extensive application at the tree level (see e.g. [4, 10 − 14]) and also at one loop (such as in [15 − 21]). This was the inspiration that kick-started the Canonical MHV Lagrangian with Mansfield's paper [22] in 2005. This demonstrated that a particular canonical transformation to the field variables of the light-cone gauge Yang-Mills resulted in another lagrangian whose vertices were Parke-Taylor MHV amplitudes, with the CSW prescription arising naturally from the gauge choice. (For a different approach to this, see ref. [23].)

The concrete form of this transformation was obtained in [24], where the Parke-Taylor form of the vertices was also shown explicitly for up to five gluons. This work was later extended in [25], which gave a prescription for dimensional regularisation of the Canonical MHV Lagrangian. It was also seen that the 'missing' (parts of) amplitudes — those that cannot be constructed using the MHV rules, such as $(++-)$ in $(2, 2)$ signature or one-loop all-+ helicity — can be recovered by realising that the field transformation itself provided extra 'completion' vertices. Although in ref. [25] this was only shown for the $(++++)$ amplitude, it is clear from the details of the calculation that together with the completion vertices all that has happened by the change of variables is that the various contributions from the light-cone Feynman diagrams have been algebraically rearranged and thus we have no doubt that all of the original light-cone Yang-Mills theory can be recovered in this way.

Field transformation approaches to deriving MHV amplitudes have also been applied to other theories. For example, in ref. [26] MHV lagrangian techniques were developed for $\mathcal{N} = 4$ supersymmetric Yang-Mills; and in ref. [27], the authors use a field transformation to make manifest the KLT relations [28] for the three- and four-graviton vertices at the action level. In ref. [20], the authors subject light-cone Yang-Mills to a holomorphic transformation with a non-trivial jacobian, and argue that this gives rise to the all-+ amplitudes. Later in [21], the Mansfield transformation is combined with a Lorentz-violating two-point one-loop counterterm which the authors argue generates the infinite sequence of one-loop all-+ scattering amplitudes. An approach complementary to the Canonical MHV Lagrangian was developed in refs. [29, 30], which makes the MHV rules manifest by continuing Yang-Mills theory to twistor space and applying a particular gauge fixing. This was extended to include massive scalars for both this approach and space-time canonical transformation techniques in [31, 32], where in the latter it was shown that the field transformations induced by these techniques are identical.

One of our main goals in this paper is to develop the ideas sketched in section 3 of [22] and build upon the work of [24]. Specifically, this is to extend the proof of a canonical MHV lagrangian for pure Yang-Mills into one for massless QCD with quarks in the fundamental

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representation — that is, to obtain a lagrangian for QCD whose Feynman rules make the MHV rules for that theory manifest. (We will also take the opportunity to fill a small gap in the original proof [22] for Yang-Mills, that appears once one realises that there are extra completion vertices that in general need to be taken into account.) Here, we thus demonstrate that the MHV rules for massless quarks have a field-theoretic origin lying within the context of our earlier work, and thus also provide the appropriate completion vertices and the starting point for dimensional regularisation and generalisation to other fields. We will also apply our methodology to the case where the quarks are massive, and use it to obtain the appropriate generalisation of the MHV rules in this scenario.

The structure of the rest of this paper is as follows. In section 2, we explain the notation and conventions we will be using. We obtain the QCD action in light-cone gauge, and then proceed to construct the field transformation that leads to the MHV lagrangian for massless QCD. The proof of this follows almost immediately [22]. We then examine the form of the MHV lagrangian in detail, giving the precise correspondence between the vertices and the MHV amplitudes. We demonstrate in section 3 that this lagrangian does indeed contain vertices corresponding to the expressions found in the literature for MHV amplitudes containing quarks for a few low-order cases. In section 4, we show how the quarks may be given masses within the MHV lagrangian mechanism. Finally in section 5 we draw our conclusions and indicate directions for future research.

2. Light-cone QCD and the transformation

In this section we will explain the conventions and notation used in this paper; perform light-cone gauge fixing of the massless QCD lagrangian; obtain the field transformation that gives the Canonical MHV Lagrangian for massless QCD; and finally make concrete the relationship between its vertices and the MHV amplitudes.

2.1 Light-cone coordinates

We follow closely the conventions laid out in [25]. The problem is adapted to a co-ordinate system defined by

$$x^0 = \tfrac{1}{\sqrt{2}}(t - x^3), \quad x^{\bar{0}} = \tfrac{1}{\sqrt{2}}(t + x^3), \quad z = \tfrac{1}{\sqrt{2}}(x^1 + ix^2), \quad \bar{z} = \tfrac{1}{\sqrt{2}}(x^1 - ix^2). \tag{2.1}$$

Note here the presence of the $1/\sqrt{2}$ factors that preserve the normalisation of the volume form. It is useful to employ a compact notation for the components of 1-forms in these co-ordinates, for which we write $(p_0, p_{\bar{0}}, p_z, p_{\bar{z}}) \equiv (\check{p}, \hat{p}, p, \bar{p})$; for momenta labelled by a number, we write that number with a decoration, for example the $n^{\rm th}$ external momentum has components $(\check{n}, \hat{n}, \tilde{n}, \bar{n})$. In these co-ordinates and with this notation, the Lorentz invariant reads

$$A \cdot B = \check{A}\,\hat{B} + \hat{A}\,\check{B} - A\bar{B} - \bar{A}B. \tag{2.2}$$

We will also make extensive use of the following bilinears:

$$(1\,2) := \hat{1}\tilde{2} - \hat{2}\tilde{1}, \quad \{1\,2\} := \hat{1}\bar{2} - \hat{2}\bar{1}. \tag{2.3}$$

These can be expressed in terms of the angle and square brackets found in the literature by considering the bispinor representation of a 4-vector p

$$p \cdot \bar{\sigma} = \sqrt{2} \begin{pmatrix} \check{p} & -p \\ -\bar{p} & \hat{p} \end{pmatrix}, \tag{2.4}$$

where we define $\sigma^\mu = (1, \boldsymbol{\sigma})$ and $\bar{\sigma}^\mu = (1, -\boldsymbol{\sigma})$, and $\boldsymbol{\sigma}$ is the 3-vector of Pauli matrices. For null p, this factorises as $(p \cdot \bar{\sigma})_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ where we can choose

$$\lambda_\alpha = 2^{1/4} \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix} \quad \text{and} \quad \tilde{\lambda}_{\dot{\alpha}} = 2^{1/4} \begin{pmatrix} -\bar{p}/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}. \tag{2.5}$$

We note here that this choice of spinors for off-shell p coincides with the CSW prescription [4] when using reference spinors $\nu = \tilde{\nu} = (2^{1/4}, 0)^{\mathrm{T}}$ [22, 24]. Hence the spinor brackets can be expressed as

$$\langle 1\, 2 \rangle := \epsilon^{\alpha\beta} \lambda_{1\alpha} \lambda_{2\beta} = \sqrt{2} \frac{(1\, 2)}{\sqrt{\hat{1}\hat{2}}} \quad \text{and} \quad [1\, 2] := \epsilon^{\dot{\alpha}\dot{\beta}} \lambda_{1\dot{\alpha}} \lambda_{2\dot{\beta}} = \sqrt{2} \frac{\{1\, 2\}}{\sqrt{\hat{1}\hat{2}}}. \tag{2.6}$$

2.2 The light-cone Dirac equation

We use the chiral Weyl representation of the Dirac matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}. \tag{2.7}$$

A Dirac spinor ψ may be decomposed into its left- and right-handed Weyl components $\bar{\omega}^{\dot{\alpha}}$ and φ_α respectively as

$$\psi = \begin{pmatrix} \bar{\omega}^{\dot{\alpha}} \\ \varphi_\alpha \end{pmatrix}. \tag{2.8}$$

For massless quarks, we solve the Dirac equation $\not{p}\psi = 0$ and obtain the following polarisation spinors:

$$\bar{u}^+(p) \equiv (\bar{\varphi}(p), 0) \quad \text{and} \quad \bar{u}^-(p) \equiv (0, \omega(p))$$

for positive- and negative-helicity *out-going* quarks, respectively; and similarly

$$v^+(p) \equiv \begin{pmatrix} \bar{\omega}(p) \\ 0 \end{pmatrix} \quad \text{and} \quad v^-(p) \equiv \begin{pmatrix} 0 \\ \varphi(p) \end{pmatrix}$$

for *out-going* antiquarks. Here we have used the following definitions for the Weyl spinors:

$$\varphi(p) = 2^{1/4} \begin{pmatrix} -p/\sqrt{\hat{p}} \\ \sqrt{\hat{p}} \end{pmatrix}, \tag{2.9}$$

$$\bar{\omega}(p) = 2^{1/4} \begin{pmatrix} \sqrt{\hat{p}} \\ \bar{p}/\sqrt{\hat{p}} \end{pmatrix}, \tag{2.10}$$

$$\bar{\varphi}(p) = 2^{1/4} (-\bar{p}/\sqrt{\hat{p}}, \sqrt{\hat{p}}), \tag{2.11}$$

$$\omega(p) = 2^{1/4} (\sqrt{\hat{p}}, p/\sqrt{\hat{p}}), \tag{2.12}$$

chosen so that external fermion states have the conventional phenomenologists' normalisation of $u^\dagger(p) u(p) = 2p^t$.

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2.3 Massless light-cone QCD

We will study a massless QCD theory with a SU($N_{\rm C}$) gauge theory. Its action is

$$S_{\rm QCD} = \int d^4x\, \bar\psi\, i\not{\mathcal{D}}\, \psi + \frac{1}{2g^2}\int d^4x\ {\rm tr}\, \mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \tag{2.13}$$

where

$$\psi = (\alpha^+, \beta^+, \beta^-, \alpha^-)^{\rm T} \quad \text{and} \quad \bar\psi = (\bar\beta^+, \bar\alpha^+, \bar\alpha^-, \bar\beta^-) \tag{2.14}$$

are the quark field and conjugate in the fundamental representation of SU($N_{\rm C}$). Note here that the superscripts $\pm$ on the spinor components denote the physical helicity for *out-going* particles, and the bar denotes a field in the conjugate representation; that $\bar\alpha^+ = (\alpha^-)^*$ should be understood, and similarly for β. We further define

$$\mathcal{F}_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu], \quad \mathcal{D}_\mu = \partial_\mu + \mathcal{A}_\mu, \quad \mathcal{A}_\mu = -\frac{ig}{\sqrt{2}} A^a_\mu T^a. \tag{2.15}$$

Our gauge group generators are normalised according to

$$[T^a, T^b] = i\sqrt{2} f^{abc} T^c, \quad {\rm tr}(T^aT^b) = \delta^{ab}. \tag{2.16}$$

As in [22, 24, 25], we quantise the theory on surfaces Σ of constant x^0, i.e. those with normal $\mu = (1,0,0,1)/\sqrt{2}$ in Minkowski co-ordinates. We choose an axial gauge $\mu\cdot\mathcal{A} = \hat{\mathcal{A}} = 0$, for which the Faddeev-Popov ghosts are completely decoupled. The lagrangian density is quadratic in the $\check{\mathcal{A}}$ field; furthermore, only the fermion field components $\bar\alpha^-$, $\bar\alpha^+$, α^- and α^+ are dynamical (inasmuch as they are the only ones to occur in the lagrangian density in terms with $\check\partial$ quantisation 'time' derivatives). Thus we integrate $\check{\mathcal{A}}$ and the remaining fermion components out of the partition function and obtain the gauge-fixed action

$$\begin{aligned} S_{\rm LCQCD} = \frac{4}{g^2}\int dx^0 (&L^{-+} + L^{-++} + L^{--+} + L^{--++} \\ &+ L^{\bar\psi\psi} + L^{\bar\psi+\psi} + L^{\bar\psi-\psi} + L^{\bar\psi+-\psi} + L^{\bar\psi\psi\bar\psi\psi}), \end{aligned} \tag{2.17}$$

where

$$L^{-+} = \ \ {\rm tr}\int_\Sigma d^3\mathbf{x}\, \bar{\mathcal{A}}(\check\partial\hat\partial - \partial\bar\partial)\mathcal{A}, \tag{2.18}$$

$$L^{-++} = -{\rm tr}\int_\Sigma d^3\mathbf{x}\, (\bar\partial\hat\partial^{-1}\mathcal{A})\, [\mathcal{A}, \hat\partial\bar{\mathcal{A}}], \tag{2.19}$$

$$L^{--+} = -{\rm tr}\int_\Sigma d^3\mathbf{x}\, [\bar{\mathcal{A}}, \hat\partial\mathcal{A}]\, (\partial\hat\partial^{-1}\bar{\mathcal{A}}), \tag{2.20}$$

$$L^{--++} = -{\rm tr}\int_\Sigma d^3\mathbf{x}\, [\bar{\mathcal{A}}, \hat\partial\mathcal{A}]\, \hat\partial^{-2}\, [\mathcal{A}, \hat\partial\bar{\mathcal{A}}] \tag{2.21}$$

is the pure Yang-Mills sector of the theory, and the terms involving the fermions are

$$L^{\bar\psi\psi} = \frac{ig^2}{\sqrt{8}}\int_\Sigma d^3\mathbf{x}\, \Big\{\bar\alpha^+(\check\partial - \omega)\alpha^- + \bar\alpha^-(\check\partial - \omega)\alpha^+\Big\}, \tag{2.22}$$

$$L^{\bar\psi+\psi} = -\frac{ig^2}{\sqrt{8}}\int_\Sigma d^3\mathbf{x}\,\Big\{\bar\alpha^+\,\bar\partial\hat\partial^{-1}(\mathcal{A}\,\alpha^-) + \bar\alpha^-\mathcal{A}\,\bar\partial\hat\partial^{-1}\alpha^+ - \bar\alpha^+(\bar\partial\hat\partial^{-1}\mathcal{A})\alpha^- - \bar\alpha^-(\bar\partial\hat\partial^{-1}\mathcal{A})\alpha^+\Big\}, \tag{2.23}$$

$$L^{\bar\psi-\psi} = -\frac{ig^2}{\sqrt{8}}\int_\Sigma d^3\mathbf{x}\,\Big\{\bar\alpha^+\bar{\mathcal{A}}\,\partial\hat\partial^{-1}\alpha^- + \bar\alpha^-\partial\hat\partial^{-1}(\bar{\mathcal{A}}\,\alpha^+) - \bar\alpha^+(\partial\hat\partial^{-1}\bar{\mathcal{A}})\alpha^- - \bar\alpha^-(\partial\hat\partial^{-1}\bar{\mathcal{A}})\alpha^+\Big\}, \tag{2.24}$$

$$\begin{aligned} L^{\bar\psi+-\psi} = -\frac{ig^2}{\sqrt{8}}\int_\Sigma d^3\mathbf{x}\,\Big\{&\bar\alpha^+\bar{\mathcal{A}}\hat\partial^{-1}(\mathcal{A}\alpha^-) + \bar\alpha^-\mathcal{A}\hat\partial^{-1}(\bar{\mathcal{A}}\alpha^+) \\ &+ \bar\alpha^+\hat\partial^{-2}(\hat\partial\mathcal{A}\bar{\mathcal{A}} - \mathcal{A}\hat\partial\bar{\mathcal{A}})\alpha^- + \bar\alpha^+\hat\partial^{-2}(\hat\partial\bar{\mathcal{A}}\mathcal{A} - \bar{\mathcal{A}}\hat\partial\mathcal{A})\alpha^- \\ &+ \bar\alpha^-\hat\partial^{-2}(\hat\partial\mathcal{A}\bar{\mathcal{A}} - \mathcal{A}\hat\partial\bar{\mathcal{A}})\alpha^+ + \bar\alpha^-\hat\partial^{-2}(\hat\partial\bar{\mathcal{A}}\mathcal{A} - \bar{\mathcal{A}}\hat\partial\mathcal{A})\alpha^+\Big\}, \end{aligned} \tag{2.25}$$

$$L^{\bar\psi\psi\bar\psi\psi} = \frac{g^4}{16}\int_\Sigma d^3\mathbf{x}\, j^a\hat\partial^{-2}j^a, \quad j^a = \sqrt{2}(\bar\alpha^+T^a\alpha^- + \bar\alpha^-T^a\alpha^+). \tag{2.26}$$

Note that in (2.22) we define the differential operator $\omega := \partial\bar\partial/\hat\partial$.

2.4 Structure of the canonical transformation

Let us now construct the field transformation that results in a MHV lagrangian for massless QCD. We label the new algebra-valued gauge fields $\mathcal{B}$ and $\bar{\mathcal{B}}$, and the new fundamental representation fermions ξ^+, $\bar\xi^-$, ξ^- and $\bar\xi^+$; their Lorentz transformation properties are the same as those of the old fields of similar decoration. We remove terms in the light-cone lagrangian with a $(-++)$ helicity structure by absorbing them into the kinetic terms of the new fields:

$$L^{-+}[\mathcal{A},\bar{\mathcal{A}}] + L^{-++}[\mathcal{A},\bar{\mathcal{A}}] + L^{\bar\psi\psi}[\alpha^\pm,\bar\alpha^\pm] + L^{\bar\psi+\psi}[\mathcal{A},\alpha^\pm,\bar\alpha^\pm] = L^{-+}[\mathcal{B},\bar{\mathcal{B}}] + L^{\bar\psi\psi}[\xi^\pm,\bar\xi^\pm]. \tag{2.27}$$

The remaining terms in the lagrangian, (2.20), (2.21) and (2.24)–(2.26) form the MHV vertices as we will show in the next section. We note that, at least classically, this choice of transformation seems sensible since it maps a field theory that is free (at the tree-level of the S-matrix) on the l.h.s. of (2.27) onto a strictly free theory in the new variables on the r.h.s. .

We note that the canonical (co-ordinate, momentum) pairs of the system (2.17) are

$$(\mathcal{A}, -\hat\partial\bar{\mathcal{A}}), \quad \left(\alpha^+, -\frac{ig^2}{\sqrt{8}}\bar\alpha^-\right) \quad \text{and} \quad \left(\alpha^-, -\frac{ig^2}{\sqrt{8}}\bar\alpha^+\right), \tag{2.28}$$

and likewise for the new fields (by replacing $\mathcal{A}\to\mathcal{B}$ and $\alpha\to\xi$ above). We have defined the momenta with respect to the lagrangian of (2.18)–(2.26). We see that up to a constant the path integral measure

$$\mathcal{D}\mathcal{A}\mathcal{D}\bar{\mathcal{A}}\,\mathcal{D}\alpha^+\mathcal{D}\bar\alpha^-\,\mathcal{D}\alpha^-\mathcal{D}\bar\alpha^+ \tag{2.29}$$

is equal to the phase space measure, and thus will be preserved if the transformation is canonical. This, and our demands on the helicity content of the resulting lagrangian, restrict the form of the transformation as follows.

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To begin, we choose $\mathcal{A}$ to be a functional of $\mathcal{B}$ alone. This, and the canonical transformation properties of a conjugate momentum, imply that

$$\hat{\partial}\bar{\mathcal{A}}^a(\mathbf{x}) = \int_\Sigma d^3\mathbf{y} \left\{ \frac{\delta \mathcal{B}^b(\mathbf{y})}{\delta \mathcal{A}^a(\mathbf{x})} \hat{\partial}\bar{\mathcal{B}}^b(\mathbf{y}) - \frac{ig^2}{\sqrt{8}} \left(\bar{\xi}^-(\mathbf{y}) \frac{\delta \xi^+(\mathbf{y})}{\delta \mathcal{A}^a(\mathbf{x})} + \bar{\xi}^+(\mathbf{y}) \frac{\delta \xi^-(\mathbf{y})}{\delta \mathcal{A}^a(\mathbf{x})} \right) \right\} \tag{2.30}$$

where all fields have the same implicit x^0 dependence. Note that we take all derivatives with respect to Grassman variables as acting from the left. By charge conservation, and the requirement that this will be a canonical transformation that results in a lagrangian whose vertices have MHV helicity content, the fermion co-ordinate transformation takes the form (which we explain below)

$$\xi^\pm(\mathbf{x}) = \int_\Sigma d^3\mathbf{y}\, R^\mp[\mathcal{A}](\mathbf{x},\mathbf{y})\, \alpha^\pm(\mathbf{y}). \tag{2.31}$$

The superscript of $R^\pm$ refers to the chirality of the Weyl spinor from which the fermion components originate: $+$ for right-handed, $-$ for left-handed. $R^\pm$ is a matrix-valued functional of $\mathcal{A}$. Putting additional factors of $\bar{\mathcal{A}}$ into the r.h.s. of (2.31) would result in terms in the resulting lagrangian with more than two fields of negative helicity; likewise with extra quark fields, since charge conservation requires these to be added in $(+-)$ helicity pairs. The behaviour of the canonical momentum under a canonical transformation is then fixed by (2.31) to be

$$\bar{\alpha}^\pm(\mathbf{x}) = \int_\Sigma d^3\mathbf{y}\, \bar{\xi}^\pm(\mathbf{y})\, R^\pm[\mathcal{A}](\mathbf{y},\mathbf{x}), \tag{2.32}$$

but it will also be useful to define the inverse transformations

$$\alpha^\pm(\mathbf{x}) = \int_\Sigma d^3\mathbf{y}\, S^\mp[\mathcal{A}](\mathbf{x},\mathbf{y})\, \xi^\pm(\mathbf{y}) \tag{2.33}$$

as well.

At this point we can immediately read off the propagators for the new fields from (2.27) as

$$\langle \mathcal{B}\bar{\mathcal{B}} \rangle = -\frac{ig^2}{2p^2} \quad \text{and} \quad \langle \xi^-\bar{\xi}^+ \rangle = \langle \xi^+\bar{\xi}^- \rangle = i\sqrt{2}\frac{\hat{p}}{p^2}. \tag{2.34}$$

By using (2.15), one obtains the canonically normalised propagator $\langle B\bar{B} \rangle = i/p^2$, and indeed in practical calculations with the MHV lagrangian it is often more convenient to absorb powers of this factor into the lagrangian's vertices and transformation series coefficients, as was done in [25]. For the purposes of this paper, however, we will account for these factors at the end of the calculations we present in the forthcoming.

If we now assume solutions for R and S as infinite series in $\mathcal{A}$, it is not hard to see that, upon substitution into the non-transformation terms of the light-cone QCD lagrangian (2.20), (2.21) and (2.24)–(2.26), this choice of transformation furnishes a set of terms with no more than two fields of negative helicity ($\bar{\mathcal{B}}$, ξ^- and $\bar{\xi}^-$) in each, but an increasing number of $\mathcal{B}$, as shown in table 1. (The number of positive-helicity quark fields present is, of course, strictly constrained by charge conservation.) Furthermore, inspection of these terms tells us that the interaction part of the MHV lagrangian should be a sum

LCQCD term	New field content
L^{--+}	$\bar{\mathcal{B}}\bar{\mathcal{B}}\mathcal{B}\cdots,\quad \bar{\xi}\xi\bar{\mathcal{B}}\mathcal{B}\cdots,\quad \bar{\xi}\xi\bar{\xi}\xi\mathcal{B}\cdots$
L^{--++}	$\bar{\mathcal{B}}\bar{\mathcal{B}}\mathcal{B}\mathcal{B}\cdots,\quad \bar{\xi}\xi\bar{\mathcal{B}}\mathcal{B}\cdots,\quad \bar{\xi}\xi\bar{\xi}\xi\mathcal{B}\mathcal{B}\cdots$
$L^{\psi-\psi}$	$\bar{\xi}\xi\bar{\mathcal{B}},\quad \bar{\xi}\xi\bar{\mathcal{B}}\mathcal{B}\cdots,\quad \bar{\xi}\xi\bar{\xi}\xi\mathcal{B}\cdots$
$L^{\bar{\psi}+-\psi}$	$\bar{\xi}\xi\bar{\mathcal{B}}\mathcal{B}\cdots,\quad \bar{\xi}\xi\bar{\xi}\xi\mathcal{B}\cdots$
$L^{\bar{\psi}\psi\bar{\psi}\psi}$	$\bar{\xi}\xi\bar{\xi}\xi,\quad \bar{\xi}\xi\bar{\xi}\xi\mathcal{B}\cdots$

Table 1: The contents of the new vertices provided by our choice of field transformation. The new fermion fields, ξ, always occur in bilinear pairs and as such $\bar{\xi}\xi$ is the sum of a term containing exactly one $-$ helicity quark, and another term with one $-$ helicity antiquark. An ellipsis $\cdots$ denotes an infinite series wherein the field to its immediate left is repeated.

of the following two terms. Switching to momentum space on the quantisation surface Σ, the first is the purely gluonic part

$$L^{\mathrm{MHV}}_{\mathrm{YM}} = \frac{1}{2}\sum_{n=3}^{\infty}\sum_{s=2}^{n}\int_{1\cdots n} V^{s}_{\mathrm{YM}}(1\cdots n)\,\mathrm{tr}(\bar{\mathcal{B}}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\bar{\mathcal{B}}_{\bar{s}}\cdots\mathcal{B}_{\bar{n}})\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^{n}\mathbf{p}_i). \tag{2.35}$$

Note here that numbered subscripts denote momentum arguments, the bar denoting negation (i.e. $\mathcal{B}_{\bar{\imath}} := \mathcal{B}(-\mathbf{p}_i)$), and our integral short-hand is defined by

$$\int_{1\cdots n} = \prod_{k=1}^{n}\frac{1}{(2\pi)^3}\int d\hat{k}\,dk\,d\bar{k}. \tag{2.36}$$

The factor of $\frac{1}{2}$ above absorbs the factor of 2 that arises from the two possible contractions of gluon external states into the trace. The second term is a new fermionic part

$$\begin{aligned} L^{\mathrm{MHV}}_{\mathrm{F}} = &\sum_{n=3}^{\infty}\sum_{s=2}^{n-1}\int_{1\cdots n}\Big\{V^{s,-+}_{\mathrm{F}}(1\cdots n)\,\bar{\xi}^{-}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\bar{\mathcal{B}}_{\bar{s}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{+}_{\bar{n}} \\ &\qquad + V^{s,+-}_{\mathrm{F}}(1\cdots n)\,\bar{\xi}^{+}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\bar{\mathcal{B}}_{\bar{s}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{-}_{\bar{n}}\Big\} \\ &+\sum_{n=4}^{\infty}\sum_{s=2}^{n-2}\int_{1\cdots n}\Big\{\frac{1}{2}V^{s,+-+-}_{\mathrm{F}}(1\cdots n)\,\bar{\xi}^{+}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{s-1}}\xi^{-}_{\bar{s}}\,\bar{\xi}^{+}_{\overline{s+1}}\mathcal{B}_{\overline{s+2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{-}_{\bar{n}} \\ &\qquad +\frac{1}{2}V^{s,-+-+}_{\mathrm{F}}(1\cdots n)\,\bar{\xi}^{-}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{s-1}}\xi^{+}_{\bar{s}}\,\bar{\xi}^{-}_{\overline{s+1}}\mathcal{B}_{\overline{s+2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{+}_{\bar{n}} \\ &\qquad +V^{s,++--}_{\mathrm{F}}(1\cdots n)\,\bar{\xi}^{+}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{s-1}}\xi^{+}_{\bar{s}}\,\bar{\xi}^{-}_{\overline{s+1}}\mathcal{B}_{\overline{s+2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{-}_{\bar{n}} \\ &\qquad +V^{s,+--+}_{\mathrm{F}}(1\cdots n)\,\bar{\xi}^{+}_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{s-1}}\xi^{-}_{\bar{s}}\,\bar{\xi}^{-}_{\overline{s+1}}\mathcal{B}_{\overline{s+2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{+}_{\bar{n}}\Big\}. \end{aligned} \tag{2.37}$$

Again, we point out the symmetry factors of $\frac{1}{2}$, and that we have absorbed the factors of $(2\pi)^3\delta^3(\sum_{i=1}^{n}\mathbf{p}_i)$ into $\int_{1\cdots n}$ for compactness.

Note that (2.35) and (2.37) have precisely the helicity and colour structure required to be identified as the interaction part of a MHV lagrangian, and thus the Feynman rules of its tree-level perturbation theory will follow the MHV rules. We will review these MHV rules and detail their precise correspondence with the MHV lagrangian in section 2.7, but we note for now that each term's unique helicity and colour structure means that it is the sole contributor to the corresponding MHV amplitude [4, 9], thus on shell the vertex must be that amplitude, up to polarisation factors. Actually, this assertion follows once we demonstrate that the extra completion vertices that follow from the transformation [25] make no contribution, and we address this in section 2.6. Off shell, each vertex is still the MHV amplitude up to polarisation factors, continued off shell by the CSW prescription. This follows essentially directly by the argument given in ref. [22], and again we sketch the proof in section 2.6.

2.5 Explicit solutions

In the meantime, let us proceed by obtaining explicit solutions to the canonical transformation. Let us begin with (2.27). We write it out explicitly, making use of (2.30), (2.31) and (2.32) to substitute for $\hat{\partial}\bar{\mathcal{A}}$, $\bar{\alpha}^{\pm}$ and $\xi^{\pm}$ respectively. For clarity, we show below only the left-handed chiral fermion components:

$$\begin{aligned}&\int_{\mathbf{xy}}\{\omega\mathcal{A}+[\mathcal{A},\zeta\mathcal{A}]\}^a(\mathbf{x})\frac{\delta\mathcal{B}^b(\mathbf{y})}{\delta\mathcal{A}^a(\mathbf{x})}\hat{\partial}\bar{\mathcal{B}}^b(\mathbf{y})-\frac{ig^2}{\sqrt{8}}\int_{\mathbf{xyz}}\{\omega\mathcal{A}+[\mathcal{A},\zeta\mathcal{A}]\}^a(\mathbf{z})\\&\times\left\{\bar{\xi}^-(\mathbf{x})\frac{\delta R^-(\mathbf{x},\mathbf{y})}{\delta\mathcal{A}^a(\mathbf{z})}\alpha^+(\mathbf{y})+\text{r.h.}\right\}+\frac{ig^2}{\sqrt{8}}\int_{\mathbf{xy}}\Big[\bar{\xi}^-(\mathbf{x})\big\{R^-(\mathbf{x},\mathbf{y})[\zeta\mathcal{A}(\mathbf{y})]\\&\qquad+\omega_yR^-(\mathbf{x},\mathbf{y})-\zeta_y[R^-(\mathbf{x},\mathbf{y})\mathcal{A}(\mathbf{y})]\big\}\alpha^+(\mathbf{y})+\text{r.h.}\Big]\\&\qquad\qquad=\int_{\mathbf{x}}\omega\mathcal{B}^a\hat{\partial}\bar{\mathcal{B}}^a-\frac{ig^2}{\sqrt{8}}\int_{\mathbf{xy}}\{\bar{\xi}^-(\mathbf{x})\omega_xR^-(\mathbf{x},\mathbf{y})\alpha^+(\mathbf{y})+\text{r.h.}\}.\end{aligned}\tag{2.38}$$

We have adopted the convenient short-hand $\int_{\mathbf{xy}\cdots}:=\int_\Sigma d^3\mathbf{x}\,d^3\mathbf{y}\cdots$, and defined the differential operator $\zeta=\bar{\partial}/\hat{\partial}$.

Now recall from [24] we obtained $\mathcal{A}$ as a series in $\mathcal{B}$:

$$\mathcal{A}_1=\sum_{n=2}^{\infty}\int_{2\cdots n}\Upsilon(1\cdots n)\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\bar{n}}\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^n\mathbf{p}_i)\tag{2.39}$$

with

$$\Upsilon(1\cdots n)=(-i)^n\frac{\hat{1}\hat{3}\cdots\widehat{n-1}}{(2\,3)\cdots(n-1,n)}.\tag{2.40}$$

This was obtained by solving equation (2.17) of [24]:

$$\int_{\mathbf{x}}\{\omega\mathcal{A}+[\mathcal{A},\zeta\mathcal{A}]\}^a(\mathbf{x})\frac{\delta\mathcal{B}^b(\mathbf{y})}{\delta\mathcal{A}^a(\mathbf{x})}=\omega\mathcal{B}^b(\mathbf{y}).\tag{2.41}$$

Thus by substituting for $\mathcal{A}$ with (2.39), we can eliminate the first terms from either side of (2.38) leaving only terms bilinear in the quark fields. Furthermore, we can use (2.41) on the l.h.s. of (2.38) to trade the $\delta R^-/\delta\mathcal{A}$ for $\delta R^-/\delta\mathcal{B}$, and thus for the left-handed parts we arrive at

$$\begin{aligned}&\int_{\mathbf{xyz}}\bar{\xi}^-(\mathbf{x})\left\{(\omega_x+\omega_y)R^-(\mathbf{x},\mathbf{y})-[\omega\mathcal{B}^a(\mathbf{z})]\frac{\delta R^-(\mathbf{x},\mathbf{y})}{\delta\mathcal{B}^a(\mathbf{z})}\right\}\alpha^+(\mathbf{y})\\&\qquad=\int_{\mathbf{xyz}}\bar{\xi}^-(\mathbf{x})\left\{\zeta_y[R^-(\mathbf{x},\mathbf{y})\mathcal{A}(\mathbf{y})]-R^-(\mathbf{x},\mathbf{y})[\zeta\mathcal{A}(\mathbf{y})]\right\}\alpha^+(\mathbf{y}).\end{aligned}\tag{2.42}$$

The same procedure yields a similar equation for the right-handed sector:

$$\int_{\mathbf{xyz}} \bar{\xi}^+(\mathbf{x}) \left\{ (\omega_x + \omega_y) R^+(\mathbf{x}, \mathbf{y}) - [\omega \mathcal{B}^a(\mathbf{z})] \frac{\delta R^+(\mathbf{x}, \mathbf{y})}{\delta \mathcal{B}^a(\mathbf{z})} \right\} \alpha^-(\mathbf{y})$$
$$= \int_{\mathbf{xyz}} \bar{\xi}^+(\mathbf{x}) \left\{ [\zeta_y R^+(\mathbf{x}, \mathbf{y})] \mathcal{A}(\mathbf{y}) - R^+(\mathbf{x}, \mathbf{y}) [\zeta \mathcal{A}(\mathbf{y})] \right\} \alpha^-(\mathbf{y}). \quad (2.43)$$

Since the quark fields are arbitrary, equations (2.42) and (2.43) determine the solution for $R^\pm$ in terms of $\mathcal{B}$. We switch to quantisation surface momentum space, and postulate a series solution of the form

$$R^\pm(12) = (2\pi)^3 \delta^3(\mathbf{p}_1 + \mathbf{p}_2) + \sum_{n=3}^{\infty} \int_{3\cdots n} R^\pm(12; 3\cdots n) \mathcal{B}_{\bar{3}} \cdots \mathcal{B}_{\bar{n}} \, (2\pi)^3 \delta^3(\textstyle\sum_{i=1}^n \mathbf{p}_i) \quad (2.44)$$

Here, momenta 1 and 2 are associated with the Fourier transforms of $\mathbf{x}$ and $\mathbf{y}$, respectively, in (2.31). For future purposes, it will often be convenient to absorb the first term above into the sum by defining $R^\pm(12;) = 1$. Writing equations (2.42) and (2.43) in momentum space and using (2.39) to substitute for $\mathcal{A}$ leads to the following two recurrence relations:

$$R^-(12; 3\cdots n)$$
$$= \frac{-i}{\omega_1 + \cdots + \omega_n} \sum_{j=2}^{n-1} \frac{\{2, P_{j+1,n}\}}{\hat{2} \, \hat{P}_{j+1,n}} R^-(1, 2{+}P_{j+1,n}; 3 \ldots j) \Upsilon(-, j{+}1, \ldots, n) \quad (2.45)$$

and

$$R^+(12; 3\cdots n)$$
$$= \frac{-i}{\omega_1 + \cdots + \omega_n} \sum_{j=2}^{n-1} \frac{\{2, P_{j+1,n}\}}{(\hat{2}{+}\hat{P}_{j+1,n}) \, \hat{P}_{j+1,n}} R^+(1, 2{+}P_{j+1,n}; 3 \ldots j) \Upsilon(-, j{+}1, \ldots, n), \quad (2.46)$$

where we define the momentum space analogue of the ω operator as $\omega_p := p\bar{p}/\hat{p}$, $P_{ij} := \sum_{k=i}^j p_k$, and $-$ as a momentum argument denotes the negative of the sum of the other arguments. We notice immediately from the above that if we put

$$R^-(12; 3\cdots n) = -\frac{\hat{1}}{\hat{2}} R^+(12; 3\cdots n) \quad (2.47)$$

into (2.45), we recover (2.46); note that this only fixes the numerator $\hat{1}$ above, whereas the sign and the denominator follow by noting that the lowest-order coefficients $R^\pm(12) = 1$ are defined for conserved momentum (i.e. at $\mathbf{p}_1 = -\mathbf{p}_2$). Thus, we need only solve for R^+.

Now one could obtain (and prove) a form for the R^+ coefficients by direct iteration of (2.46) and induction on n, but in fact it turns out that the recurrence relation (2.46) is nothing other than a re-labelling of the one used in ref. [24] used to obtain the Υ coefficients,

$$\Upsilon(1\cdots n) = \frac{-i}{\omega_1 + \cdots + \omega_n} \sum_{j=2}^{n-1} \left(\frac{\bar{P}_{j+1,n}}{\hat{P}_{j+1,n}} - \frac{\bar{P}_{2j}}{\hat{P}_{2j}} \right) \Upsilon(-, 2, \ldots, j) \Upsilon(-, j{+}1, \ldots, n), \quad (2.48)$$

the solution to which was proved to be (2.40). If we now put

$$R^+(12;3\cdots n)=\Upsilon(213\cdots n)=(-i)^n\frac{\hat{2}\hat{3}\cdots\widehat{n-1}}{(1\,3)(3\,4)\cdots(n-1,n)} \tag{2.49}$$

into (2.46) (and swap momenta 1 and 2), we arrive at (2.48), and (2.49) is thereby proved.

The inverse fermion transformation, $S^\pm$, may be obtained from $R^\pm$ in an order-by-order manner. Let us begin by writing it as

$$S^\pm(12)=\sum_{n=2}^{\infty}\int_{3\cdots n}S^\pm(12;3\cdots n)\mathcal{B}_{\bar{3}}\cdots\mathcal{B}_{\bar{n}}\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^n\mathbf{p}_i), \tag{2.50}$$

where momenta 1 and 2 correspond to the Fourier transforms of $\mathbf{x}$ and $\mathbf{y}$ in (2.33), and we have absorbed the $\mathcal{O}(\mathcal{B}^0)$ (i.e. $n=2$) term into the sum by defining $S^\pm(12;)=1$. The $S^\pm$ coefficients satisfy the recurrence relations

$$S^\pm(12;3\cdots n)=-\sum_{j=2}^{n-1}S^\pm(1,-;3\ldots j)R^\pm(-,2;j{+}1,\ldots,n). \tag{2.51}$$

Now it is clear from (2.47) that

$$S^+(12;3\cdots n)=-\frac{\hat{2}}{\hat{1}}S^-(12;3\cdots n) \tag{2.52}$$

where again the overall normalisation is fixed by the lowest order coefficient. We state that (2.51) is solved by

$$S^-(12;3\cdots n)=(-i)^n\frac{\hat{1}\hat{4}\cdots\hat{n}}{(3\,4)\cdots(n-1,n)(n\,2)}=\Upsilon(13\cdots n2) \tag{2.53}$$

where in the case of $S^-(12;3)$ only the first factor in the numerator and the last factor in the denominator are retained. The proof is by induction on n. It is easy to check the initial step by direct iteration of (2.51) for $n=3,4$, and the inductive step is given in appendix A.1.[1]

Finally, let us solve for $\bar{\mathcal{A}}$. That the transformation is canonical requires that

$$\begin{aligned}\int_{\mathbf{x}}\Big\{\mathrm{tr}\,\check{\partial}\mathcal{A}\hat{\partial}\bar{\mathcal{A}}-\frac{ig^2}{\sqrt{8}}\int d^3\mathbf{x}(\bar{\alpha}^-\check{\partial}\alpha^++\bar{\alpha}^+\check{\partial}\alpha^-)\Big\}\\=\int_{\mathbf{x}}\Big\{\mathrm{tr}\,\check{\partial}\mathcal{B}\hat{\partial}\bar{\mathcal{B}}-\frac{ig^2}{\sqrt{8}}\int d^3\mathbf{x}(\bar{\xi}^-\check{\partial}\xi^++\bar{\xi}^+\check{\partial}\xi^-)\Big\}\end{aligned} \tag{2.54}$$

Now consider the functional form of $\hat{\partial}\bar{\mathcal{A}}$ as given by (2.30). We can split it into two pieces,

$$\hat{\partial}\bar{\mathcal{A}}=\hat{\partial}\bar{\mathcal{A}}^0+\hat{\partial}\bar{\mathcal{A}}^{\mathrm{F}}, \tag{2.55}$$

[1] We could, of course, have obtained $S^\pm$ in a manner similar to $R^\pm$, by deriving the recurrence relations and using (2.53) to map to the Υ recurrence relation.

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where the first term depends only on $\mathcal{B}$ and $\bar{\mathcal{B}}$, and the second contains the fermion dependence. If we substitute this into (2.54), we see that we must have

$$\operatorname{tr}\int_{\mathbf{x}} \check{\partial}\mathcal{A}\hat{\partial}\bar{\mathcal{A}}^0 = \operatorname{tr}\int_{\mathbf{x}} \check{\partial}\mathcal{B}\hat{\partial}\bar{\mathcal{B}}, \tag{2.56}$$

similar to the case of ref. [24], but in $\bar{\mathcal{A}}^0$ instead of $\bar{\mathcal{A}}$. Thus the first part of (2.30) is taken care of if we use the pure-gauge solution for $\bar{\mathcal{A}}^0$ found in [24]. In momentum space this is

$$\bar{\mathcal{A}}^0_{\bar{1}} = \sum_{m=2}^{\infty}\sum_{s=2}^{m}\int_{2\cdots m}\frac{\hat{s}^2}{\hat{1}^2}\Upsilon(1\cdots m)\mathcal{B}_{\bar{2}}\cdots\bar{\mathcal{B}}_{\bar{s}}\cdots\mathcal{B}_{\bar{m}}\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^m\mathbf{p}_i). \tag{2.57}$$

From this, we immediately see that the pure-gauge MHV lagrangian of [24, 22] is recovered via the terms that $\bar{\mathcal{A}}^0$ contributes to $\bar{\mathcal{A}}$ when used in L^{--+} and L^{--++}.

In quantisation surface momentum space, what is left of (2.54) is

$$-\operatorname{tr}\check{\partial}\mathcal{A}_1\hat{1}\bar{\mathcal{A}}^{\mathrm{F}}_{\bar{1}} + \frac{g^2}{\sqrt{8}}(\bar{\alpha}^-_1\check{\partial}\alpha^+_{\bar{1}} + \bar{\alpha}^+_1\check{\partial}\alpha^-_{\bar{1}}) = \frac{g^2}{\sqrt{8}}(\bar{\xi}^-_1\check{\partial}\xi^+_{\bar{1}} + \bar{\xi}^+_1\check{\partial}\xi^-_{\bar{1}}). \tag{2.58}$$

The form of (2.30) tells us that $\bar{\mathcal{A}}^{\mathrm{F}}$ can be expanded as

$$\begin{aligned}\bar{\mathcal{A}}^{\mathrm{F}}_{\bar{1}} = -\frac{g^2}{\hat{1}\sqrt{8}}\sum_{\pm}\sum_{n=3}^{\infty}\sum_{j=1}^{n-2}\int_{2\cdots n} K^{\pm(j)}(1\cdots n)\Big\{&\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\bar{j}}\xi^{\mp}_{\overline{j+1}}\bar{\xi}^{\pm}_{\overline{j+2}}\mathcal{B}_{\overline{j+3}}\cdots\mathcal{B}_{\bar{n}}\\ &+\frac{1}{N_{\mathrm{C}}}\bar{\xi}^{\pm}_{\overline{j+2}}\mathcal{B}_{\overline{j+3}}\cdots\mathcal{B}_{\bar{n}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\bar{j}}\xi^{\mp}_{\overline{j+1}}\Big\}(2\pi)^3\delta^3(\textstyle\sum_{i=1}^n\mathbf{p}_i)\end{aligned} \tag{2.59}$$

We substitute the expansions for $\mathcal{A}$, $\bar{\mathcal{A}}^{\mathrm{F}}$ and $\alpha^\pm$ into (2.58) and, after some careful relabelling, match the coefficients of the strings of fields on either side. The result is the following recurrence relation for the $K^\pm$:[2]

$$\begin{aligned}K^{\pm(j)}(1\cdots n) = &-\sum_{r=2}^{j}\Upsilon(-,1,2,\ldots,r)K^{\pm(j-r+1)}(-,r+1,\ldots,n)\\ &-\sum_{l=j+2}^{n-1}\sum_{r=1}^{j}\Upsilon(-,l+1,\ldots,n,1,2,\ldots,r)K^{\pm(j-r+1)}(-,r+1,\ldots,l)\\ &-\sum_{l=j+2}^{n}R^\pm(j+2,-;j+3,\ldots,l)S^\pm(-,j+1;l+1,\ldots,n,1,2,\ldots,j),\end{aligned} \tag{2.60}$$

Note that in the case where a sum's upper limit is less than the lower limit, the sum is taken to vanish. Let us solve for the first few coefficients. For $n=3$, $j=1$, only the third term in (2.60) contributes:

$$K^{-(1)}(123) = -S^-(32;1) = -i\frac{\hat{3}}{(2\,3)}, \tag{2.61}$$

[2]Note that in the interest of clarity and to emphasise the origin of the relevant terms, we have *not* substituted for $R^\pm$ and $S^\pm$ in terms of Υ in the forthcoming.

and for $n = 4$, $j = 1$, only the second and third term contribute:

$$\begin{aligned} K^{-(1)}(1234) &= -\Upsilon(-,4,1)K^{-(1)}(-,2,3) - S^{-}(32;41) - R^{-}(3,-;4)S^{-}(-,2;1) \\ &= -\frac{\hat{3}^2}{(2\,3)(3\,4)}. \end{aligned} \tag{2.62}$$

Similarly, we can obtain next few coefficients:

$$K^{-(2)}(1234) = -\frac{\hat{3}\hat{4}}{(2\,3)(3\,4)}, \tag{2.63}$$

$$K^{-(1)}(12345) = i\frac{\hat{3}^2\hat{4}}{(2\,3)(3\,4)(4\,5)}, \tag{2.64}$$

$$K^{-(2)}(12345) = i\frac{\hat{3}\hat{4}^2}{(2\,3)(3\,4)(4\,5)}, \tag{2.65}$$

$$K^{-(3)}(12345) = i\frac{\hat{3}\hat{4}\hat{5}}{(2\,3)(3\,4)(4\,5)}, \tag{2.66}$$

and so-on, from which we propose

$$K^{-(j)}(1\cdots n) = -(-i)^n \frac{\widehat{j+2}\,\hat{3}\hat{4}\cdots\widehat{n-1}}{(2\,3)(3\,4)\cdots(n{-}1,n)} = -\frac{\widehat{j+2}}{\hat{1}}\Upsilon(1\cdots n). \tag{2.67}$$

This can be proved by induction on n. The foregoing obviously furnishes the initial step, and the inductive part follows upon substituting (2.67) into (2.60) and evaluating the sums. This is a straightforward but tedious calculation, which we sketch in appendix A.2.

We could derive K^+ by a similar process, but there is a short-cut. Notice that it satisfies the same recurrence relation as K^-, except for the last term in (2.60). For this last term, we observe that

$$\begin{aligned} &R^{+}(j+2,-;j+3,\ldots,l)S^{+}(-,j+1;l+1,\ldots,n,1,\ldots,j) = \\ &\qquad -\frac{\widehat{j+1}}{\widehat{j+2}}R^{-}(j+2,-;j+3,\ldots,l)S^{-}(-,j+1;l+1,\ldots,n,1,\ldots,j), \end{aligned} \tag{2.68}$$

so it is easy to see that

$$K^{+(j)}(12\cdots n) = -\frac{\widehat{j+1}}{\widehat{j+2}}K^{-(j)}(12\cdots n). \tag{2.69}$$

To complete this subsection, we illustrate the new completion vertices involving fermions arising from this transformation in figure 1. These vertices augment the purely gluonic completion vertices found in ref. [25].

2.6 MHV vertices: an indirect proof

Let us now return to the discussion begun at the end of section 2.4 — specifically, we complete the proof that the resulting vertices in the transformed lagrangian are MHV vertices, i.e. are precisely the MHV amplitudes up to polarisation factors, continued off shell by the CSW prescription.

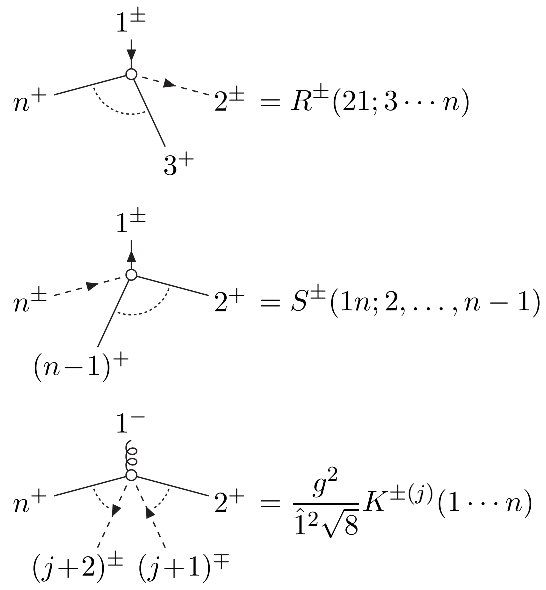

Figure 1: The non-trivial MHV completion vertices for massless QCD that involve fermions. Curly and solid, arrowless lines are $\mathcal{A}$ and $\mathcal{B}$, respectively; solid lines with arrows represent correlation function insertions of $\alpha^{\pm}$ ($\bar{\alpha}^{\pm}$) when the arrow points outwards (inwards), and $\xi^{\pm}$ ($\bar{\xi}^{\pm}$) attach to the dotted lines with the arrows pointing inwards (outwards). The direction of the arrow shows charge flow in both cases, and the missing lines are for +-helicity gluons. (All momenta are out-going.)

We already saw at the end of section 2.4 that on shell these vertices must be proportional to MHV amplitudes, if we can ignore the completion vertices. The completion vertices are just the expansion coefficients Υ, $K^{\pm}$, $R^{\pm}$ and $S^{\pm}$, examples of which are given in figure 1. These are indeed involved in the tree level contributions to the *full* off-shell MHV amplitudes (i.e. the ones that would then coincide off shell with the expression from light-cone QCD [25]). Since by LSZ reduction, the completion vertices are multiplied by p_1^2 (where p_1 is the momentum of the field being expanded), which then vanishes in the on-shell limit, the only way these contributions can survive the on-shell limit is if the expansion coefficients themselves diverge in this limit. From their explicit structure however, it is immediately clear that for non-vanishing momenta they only have collinear divergences, and thus for generic values, they make no contribution. (In the kinematically special case of the three-point vertex with all legs external and on shell, the momenta are forced to be collinear. The three-point completion vertices yield the three-point $\overline{\text{MHV}}$ amplitude, as covered in section 3.4.)

The remaining question to be settled is whether in the transformed lagrangian, the vertices can have additional terms that vanish on shell. We first note, as in ref. [22], that since the transformation is at equal light-cone time x^0 and neither the remaining vertices of the light-cone lagrangian (2.20), (2.21) and (2.24)–(2.26) nor the transformation, contain $\check{\partial}$, the resulting vertices must take the form $\delta(\sum_i \check{p}_i)V$, where V only depends on the other components of momenta. It follows that V has no way to tell whether the on-shell condition

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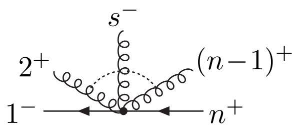
$\check{p}_i = p_i\bar{p}_i/\hat{p}_i$ is satisfied for its momenta. V may however have a term which on shell reduces to $\sum_i \check{p}_i$ and thus vanishes, i.e. have a term of the form $\sum_i p_i\bar{p}_i/\hat{p}_i$. Mansfield proves for Yang-Mills that the vertices are holomorphic, which trivially rules out such a term except in the kinematically special case of the three-point vertex [22]. The proof could be readily extended to this case, however it is more direct simply to note that the remaining vertices of the light-cone lagrangian and the expansion coefficients are explicitly holomorphic, and thus it is immediate that the expressions for V are also holomorphic. Since it is already clear that the special case of the three-point vertices in the transformed lagrangian are MHV vertices, this completes the proof that all the vertices in the transformed lagrangian are MHV vertices.

2.7 The precise MHV rules/MHV lagrangian correspondence

Let us now establish the precise relationship between our lagrangian's vertices and the MHV amplitudes with matching particle and helicity content. We will need to know this in order to check that the Feynman rules of the MHV lagrangian we have constructed match the MHV rules for QCD with massless quarks.

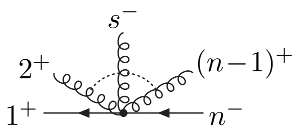
First, let us review the MHV rules for massless quarks in the fundamental representation, given in ref. [9] (for the $\mathrm{U}(N_\mathrm{C})$ case). These are as one might expect: the propagator is the scalar propagator, i/p^2, and the off-shell prescription is as in the purely gluonic case, but one adds two new classes of vertices (using the expressions for the corresponding amplitudes found in ref. [33], here adapted to our conventions for normalisation and fermion operator orderings) to the fold. The first class of MHV vertices with massless quarks has either the quark or antiquark carrying a negative helicity (by conservation of helicity) and one gluon of negative helicity. Thus they are



$$= A(1_\mathrm{q}^-, 2^+, \dots, s^-, \dots, (n-1)^+, n_{\bar{\mathrm{q}}}^+) \qquad (2.70)$$

$$= ig^{n-2}\frac{\langle 1\, s\rangle^3\langle s\, n\rangle}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n-1, n\rangle\langle n\, 1\rangle},$$

and



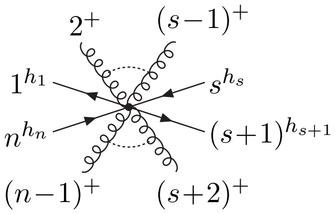
$$= A(1_\mathrm{q}^+, 2^+, \dots, s^-, \dots, (n-1)^+, n_{\bar{\mathrm{q}}}^-) \qquad (2.71)$$

$$= ig^{n-2}\frac{\langle 1\, s\rangle\langle n\, s\rangle^3}{\langle 1\, 2\rangle\langle 2\, 3\rangle\cdots\langle n-1, n\rangle\langle n\, 1\rangle}.$$

Treated as amplitudes, these would be associated with the colour structure

$$(T^{a_2}\cdots T^{a_{n-1}})_{i_1}{}^{\bar{\imath}_n} \qquad (2.72)$$

in the colour-ordered decomposition. (We remind the reader that the full amplitude here would be obtained by summing over permutations of the gluon labels $2, \dots, n-1$.) The second class is the MHV vertices with two quark-antiquark pairs. By conservation of helicity on the quark lines, these must have all gluons with positive helicity in order to be of the MHV helicity content. Diagrammatically, these vertices are



$$= A(1_{\rm q}^{h_1} 2^+ \cdots s_{\bar{\rm q}}^{h_s} (s+1)_{\rm q}^{h_{s+1}} \cdots n_{\bar{\rm q}}^{h_n}), \tag{2.73}$$

and they associate with the colour structure

$$(T^{a_2} \cdots T^{a_{s-1}})_{i_1}{}^{\bar{\imath}_s} (T^{a_{s+2}} \cdots T^{a_{n-1}})_{i_{s+1}}{}^{\bar{\imath}_n}, \tag{2.74}$$

it being understood that when $s = 2$ and/or $s = n-2$ that the first and/or second T-string respectively is the identity matrix, and the partial amplitudes are defined by

$$\begin{aligned} A(1_{\rm q}^+ 2^+ \cdots s_{\bar{\rm q}}^- s{+}1_{\rm q}^+ \cdots n_{\bar{\rm q}}^-) &= ig^{n-2} \frac{\langle s\, n\rangle^2}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \\ &\qquad \times \left(\langle 1\, s\rangle\langle s{+}1, n\rangle + \frac{1}{N_{\rm C}} \langle s, s{+}1\rangle\langle n\, 1\rangle \right), \end{aligned} \tag{2.75}$$

$$\begin{aligned} A(1_{\rm q}^- 2^+ \cdots s_{\bar{\rm q}}^+ s{+}1_{\rm q}^- \cdots n_{\bar{\rm q}}^+) &= ig^{n-2} \frac{\langle 1, s{+}1\rangle^2}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle} \\ &\qquad \times \left(\langle 1\, s\rangle\langle s{+}1, n\rangle + \frac{1}{N_{\rm C}} \langle s, s+1\rangle\langle n\, 1\rangle \right), \end{aligned} \tag{2.76}$$

$$A(1_{\rm q}^+ 2^+ \cdots s_{\bar{\rm q}}^+ s{+}1_{\rm q}^- \cdots n_{\bar{\rm q}}^-) = -ig^{n-2} \frac{\langle 1\, s\rangle\langle s{+}1, n\rangle^3}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n\, 1\rangle}\,, \tag{2.77}$$

$$A(1_{\rm q}^+ 2^+ \cdots s_{\bar{\rm q}}^- s{+}1_{\rm q}^- \cdots n_{\bar{\rm q}}^+) = -i \frac{g^{n-2}}{N_{\rm C}} \frac{\langle s, s{+}1\rangle^3}{\langle 1\, 2\rangle\langle 2\, 3\rangle \cdots \langle n{-}1, n\rangle}\,. \tag{2.78}$$

Note that these are the four independent helicity configurations for the quarks permitted by helicity conservation along the quark lines (when the amplitudes are decomposed in this manner, *cf.* [33] which uses a slightly different arrangement of the terms). There are no MHV vertices with more quark-antiquark lines.

Now, the terms of (2.37) clearly have colour structures of the form (2.72) and (2.74), and as we noted at the end of section 2.4, each MHV amplitude has contributions from only one vertex in the lagrangian, so at tree-level there is a one-to-one correspondence between the vertices V of the MHV lagrangian and the partial amplitudes (2.70), (2.71) and (2.75)–(2.78). In order to test this correspondence, we must know the external state polarisation factors. The gluon polarisation vectors are given in the spinor helicity formalism by

$$E_+ = \sqrt{2} \frac{\nu \tilde{\lambda}}{\langle \nu\, \lambda\rangle} \quad \text{and} \quad E_- = \sqrt{2} \frac{\lambda \tilde{\nu}}{[\nu\, \lambda]} \tag{2.79}$$

where $\nu\tilde{\nu} = \mu$, the null vector normal to the quantisation surface Σ. Then in co-ordinates $E_+ = \bar{E}_- = -1$ so by the LSZ theorem, when an external $+$ $(-)$ polarisation state is contracted into a $\mathcal{A}$ ($\bar{\mathcal{A}}$) vertex from the lagrangian, it contributes a factor of

$$-1 \times -\frac{ig}{\sqrt{2}}, \tag{2.80}$$

where the second factor restores the canonical normalisation of the gauge field from (2.15). This means that the correspondence between the vertex $V^s_{\rm YM}(1\cdots n)$ and the pure-gluon partial amplitude $A(1^-2^+\cdots s^-\cdots n^+)$ proceeds according to:

$$\begin{aligned}\frac{4i}{g^2}\times\left(-1\times-\frac{ig}{\sqrt{2}}\right)^n \times V^s_{\rm YM}(1\cdots n) &= ig^{n-2}\frac{\langle 1\, s\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle} \\ &= A(1^-2^+\cdots s^-\cdots n^+),\end{aligned}$$

that is,

$$V^s_{\rm YM}(1\cdots n) = (-i\sqrt{2})^{n-4}\frac{\langle 1\, s\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle},$$

which is the expression for $V^s_{\rm YM}(1\cdots n)$ given in [24].[3] (Of course, this follows immediately following the discussion below (2.57).)

For the quark correspondence, let us consider first the LSZ reduction *before* we remove the non-dynamical fermionic degrees of freedom. For example, in this context, an out-going + helicity quark with momentum p produces a term

$$\bar{\varphi}(p)_{\dot\alpha}(-ip\cdot\sigma)^{\dot\alpha\beta}\langle\cdots\begin{pmatrix}\beta^-\\ \alpha^-\end{pmatrix}_\beta\cdots\rangle, \tag{2.81}$$

where $\bar{\varphi}(p)$ is the appropriate polarisation spinor of (2.11), and $-ip\cdot\sigma$ is the inverse of the propagator obtained from (2.13). We can now compute the correlation function using the partition function furnished by the MHV lagrangian. Since $\varphi_1 \equiv \beta^-$ is replaced by its equation of motion, one might expect this non-propagating component to complicate things. Thankfully

$$\bar{\varphi}(p)(-ip\cdot\sigma) = -\frac{i}{2^{1/4}\sqrt{\hat{p}}}(0,p^2), \tag{2.82}$$

so it does not arise in the computation. We proceed to replace $\varphi_2 \equiv \alpha^-$ with its expression in terms of the new variables, and note that momentum conservation implies only the leading order term ξ^- survives the on-shell limit for generic momenta (i.e., we note that the S-matrix equivalence theorem applies here). The propagator $\langle\xi^-\bar{\xi}^+\rangle$ of (2.34) cancels factors in (2.82) to leave a polarisation factor

$$2^{1/4}\sqrt{\hat{p}}. \tag{2.83}$$

One may show similarly that the same expression applies for the − helicity state, and for the antiquarks. In summary, we state the polarisation spinors and the fields in the lagrangian associated with each out-going state in table 2. We note here that this accounts for the use of a *scalar* propagator in the MHV rules for fermions, rather than the fermion propagators we gave in (2.34) which joins the vertices of the MHV lagrangian. In the

State		Polarisation	Field
particle	$+$	$\bar{\varphi}(p)$	$\bar{\xi}^+$
	$-$	$\omega(p)$	$\bar{\xi}^-$
antiparticle	$+$	$\bar{\omega}(p)$	ξ^+
	$-$	$\varphi(p)$	ξ^-

Table 2: The polarisation spinor and lagrangian field associated with each out-going quark state.

former the numerator $\sqrt{2}\hat{p}$ of (2.34) has been absorbed into the vertices at either end as the polarisation factors to yield the familiar MHV amplitudes.

Now consider the MHV amplitude with one quark-antiquark pair. We will choose its external state to be

$$\langle 0| q_1^{\pm} A_2^+ \cdots A_s^- \cdots A_{n-1}^+ \bar{q}_n^{\mp},$$

where $\langle 0|$ is the (asymptotic) free vacuum. This contracts into one of the vertices in the first term of (2.37) (depending on the quarks' helicities), and multiplies it by an external state factor of

$$(-1) \times \frac{4i}{g^2} \times \left(\frac{ig}{\sqrt{2}}\right)^{n-2} \times 2^{1/4}\sqrt{\hat{1}} \times 2^{1/4}\sqrt{\hat{n}}.$$

Considering the factors delimited by $\times$ symbols, the first comes from Fermi statistics; the second from the path integral; the third from gluon polarisation and normalisation; and the final two from the external state spinors (see above). Thus we have

$$V_{\mathrm{F}}^{s,\pm\mp}(1\cdots n) = \frac{2^{(n-7)/2} g^{4-n}}{i^{n+1}\sqrt{\hat{1}}\sqrt{\hat{n}}} A(1_{\mathrm{q}}^{\pm}, 2^+, \ldots, s^-, \ldots, (n-1)^+, n_{\bar{\mathrm{q}}}^{\mp}); \tag{2.84}$$

concretely,

$$V_{\mathrm{F}}^{s,+-}(1\cdots n) = \frac{g^2}{i^n\sqrt{8}} \frac{\hat{2}\cdots\widehat{n-1}\,(1\,s)(n\,s)^3}{\hat{s}^2\hat{n}\,(1\,2)\cdots(n-1,n)(n\,1)}, \tag{2.85}$$

$$V_{\mathrm{F}}^{s,-+}(1\cdots n) = \frac{g^2}{i^n\sqrt{8}} \frac{\hat{2}\cdots\widehat{n-1}\,(1\,s)^3(s\,n)}{\hat{1}\hat{s}^2\,(1\,2)\cdots(n-1,n)(n\,1)}, \tag{2.86}$$

using (2.6). For vertices with two quark-antiquark pairs, we choose the external state

$$\langle 0| q_1^{h_1} A_2^+ \cdots A_{s-1}^+ \bar{q}_s^{h_s}\, q_{s+1}^{h_{s+1}} A_{s+2}^+ \cdots A_{n-1}^+ \bar{q}_n^{h_n}.$$

By contracting into the second term of (2.37) as appropriate to the helicity content, we arrive at

$$V_{\mathrm{F}}^{s,h_1 h_s h_{s+1} h_n}(1\cdots n) = \frac{2^{n/2-5} g^{6-n}}{i^{n+1}\sqrt{\hat{1}}\sqrt{\hat{s}}\sqrt{\widehat{s+1}}\sqrt{\hat{n}}} A(1_{\mathrm{q}}^{h_1} 2^+ \cdots s_{\bar{\mathrm{q}}}^{h_s} (s+1)_{\mathrm{q}}^{h_{s+1}} \cdots n_{\bar{\mathrm{q}}}^{h_n}), \tag{2.87}$$

where the A on the r.h.s. are given by (2.75)–(2.78). Note that our arrangement of the colour structure has combined the terms such that $V_{\mathrm{F}}^{s,+-+-}$ and $V_{\mathrm{F}}^{s,-+-+}$ have both $\mathcal{O}(1)$ and $\mathcal{O}(1/N_{\mathrm{C}})$ pieces, whereas $V_{\mathrm{F}}^{s,++--}$ has only a $\mathcal{O}(1)$ piece and $V_{\mathrm{F}}^{s,+--+}$ has only a $\mathcal{O}(1/N_{\mathrm{C}})$ piece.

[3] Adjusted for the normalisation in use here.

3. Example vertices

In this section, we will give some explicit examples of the terms in our MHV lagrangian, and verify explicitly that these terms are proportional off shell to the known tree-level MHV partial amplitudes up to external polarisation factors (i.e. that they satisfy (2.85)–(2.87)).

3.1 Two quarks and two gluons

Let us begin by considering the vertex $V_{\mathrm{F}}^{3,+-}(1234)$ that couples two gluons of opposing helicities to a quark-antiquark pair, i.e. the sole contributor to $A(1_{\mathrm{q}}^{+}2^{+}3^{-}4_{\bar{\mathrm{q}}}^{-})$. Looking at table 1, we see that, based upon field content, the term we are considering receives contributions from L^{--+}, $L^{\bar{\psi}+-\psi}$ and $L^{\bar{\psi}-\psi}$ (in (2.20), (2.25) and (2.24) respectively). Let us consider each in turn. Written in momentum space, (2.20) is

$$L^{--+} = i\,\mathrm{tr}\int_{123} \frac{\hat{3}}{\hat{1}\hat{2}}(1\,2)\,\bar{\mathcal{A}}_{\bar{1}}\bar{\mathcal{A}}_{\bar{2}}\mathcal{A}_{\bar{3}}, \tag{3.1}$$

where here and in the foregoing, a momentum-conserving δ function of the sum of all the momenta in the integral measure is omitted for clarity. The term with the relevant colour structure is extracted by substituting for the second $\bar{\mathcal{A}}$ with the lowest-order $\bar{\xi}^{+}\xi^{-}$ term in (2.59). Carefully relabelling, this term is

$$-\frac{g^2}{\sqrt{8}}\int_{1234} \frac{\hat{2}\hat{4}}{\hat{3}(\hat{1}+\hat{4})^2}\frac{(3\,2)}{(4\,1)}\,\bar{\xi}_{\bar{1}}^{+}\mathcal{B}_{\bar{2}}\bar{\mathcal{B}}_{\bar{3}}\xi_{\bar{4}}^{-}. \tag{3.2}$$

Next, consider the contribution from $L^{\bar{\psi}+-\psi}$: here, only the leading order substitution are needed for the fields involved so we simply extract the relevant term from the momentum-space representation of (2.25), giving

$$\frac{g^2}{\sqrt{8}}\int_{1234} \frac{\hat{3}-\hat{2}}{(\hat{1}+\hat{4})^2}\,\bar{\xi}_{\bar{1}}^{+}\mathcal{B}_{\bar{2}}\bar{\mathcal{B}}_{\bar{3}}\xi_{\bar{4}}^{-}. \tag{3.3}$$

Finally $L^{\bar{\psi}-\psi}$, which in momentum space is

$$L^{\bar{\psi}-\psi} = -\frac{ig^2}{\sqrt{8}}\int_{123}\left\{\left(\frac{\tilde{3}}{\hat{3}}-\frac{\tilde{2}}{\hat{2}}\right)\bar{\alpha}_{\bar{1}}^{+}\bar{\mathcal{A}}_{\bar{2}}\alpha_{\bar{3}}^{-} + \left(\frac{\tilde{2}+\tilde{3}}{\hat{2}+\hat{3}}-\frac{\tilde{2}}{\hat{2}}\right)\bar{\alpha}_{\bar{1}}^{-}\bar{\mathcal{A}}_{\bar{2}}\alpha_{\bar{3}}^{+}\right\}, \tag{3.4}$$

contributes two terms from the next-to-leading order substitutions for $\bar{\alpha}^{+}$ and $\bar{\mathcal{A}}$ from (2.33) and (2.57), leading to

$$\frac{g^2}{\sqrt{8}}\int_{1234}\left\{\frac{\hat{1}+\hat{2}}{\hat{3}\hat{4}}\frac{(4\,3)}{(1\,2)} + \frac{\hat{3}^2}{\hat{4}(\hat{1}+\hat{4})^2}\frac{(1\,4)}{(2\,3)}\right\}\bar{\xi}_{\bar{1}}^{+}\mathcal{B}_{\bar{2}}\bar{\mathcal{B}}_{\bar{3}}\xi_{\bar{4}}^{-}. \tag{3.5}$$

Summing over the coefficients, accounting for conservation of momentum and the normalization of $\mathcal{B}$, and comparing with (2.37), one can read off the MHV vertex:

$$V_{\mathrm{F}}^{3,+-}(1234) = -\frac{g^2}{\sqrt{8}}\frac{\hat{2}}{\hat{3}\hat{4}}\frac{(1\,3)(4\,3)^2}{(1\,2)(2\,3)(4\,1)}, \tag{3.6}$$

and it is immediate to see that this satisfies (2.84).

One may treat the remaining vertices similarly, by considering the other possible choices of substitution for $\bar{\mathcal{A}}$. We reproduce these vertices below:

$$\begin{aligned}
V_{\mathrm{F}}^{3,-+}(1234) &= \frac{g^2}{\sqrt{8}}\frac{\hat{2}}{\hat{1}\hat{3}}\frac{(1\,3)^3}{(1\,2)(2\,3)(4\,1)},\\
V_{\mathrm{F}}^{2,+-}(1234) &= -\frac{g^2}{\sqrt{8}}\frac{\hat{3}}{\hat{2}\hat{4}}\frac{(2\,4)^3}{(2\,3)(3\,4)(4\,1)},\\
V_{\mathrm{F}}^{2,-+}(1234) &= \frac{g^2}{\sqrt{8}}\frac{\hat{3}}{\hat{1}\hat{2}}\frac{(1\,2)^2(2\,4)}{(2\,3)(3\,4)(4\,1)}.
\end{aligned}$$

It is easy to see that these satisfy (2.85) and (2.86).

3.2 Four quarks

In amplitudes with two or more quark-antiquark pairs, terms $\mathcal{O}(1/N_{\mathrm{C}})$ and higher contribute at the tree-level through the sub-leading colour structures, and we must retain these terms throughout the analysis. These terms are generated by the $\mathrm{SU}(N_{\mathrm{C}})$ Fierz identity

$$(T^a)_i^{\bar{j}}(T^a)_k^{\bar{l}} = \delta_i^{\bar{l}}\delta_k^{\bar{j}} - \frac{1}{N_{\mathrm{C}}}\delta_i^{\bar{j}}\delta_k^{\bar{l}},$$

where second term on the r.h.s. comes about because the generators of $\mathrm{SU}(N_{\mathrm{C}})$ are traceless.[4] We note that the four quark vertices $V_{\mathrm{F}}^{s,h_1h_2h_3h_4}(1234)$ receive contributions from $L^{\bar{\psi}-\psi}$ and $L^{\bar{\psi}\psi\bar{\psi}\psi}$. In momentum space, these are given by (3.4) and

$$\begin{aligned}
L^{\bar{\psi}\psi\bar{\psi}\psi} = \frac{g^4}{8}\int_{1234}\Bigg\{&\left(\frac{1}{(\hat{1}+\hat{4})^2}+\frac{1}{N_{\mathrm{C}}}\frac{1}{(\hat{1}+\hat{2})^2}\right)(\bar{\alpha}_{\bar{1}}^-\alpha_{\bar{2}}^+\bar{\alpha}_{\bar{3}}^-\alpha_{\bar{4}}^+ + \bar{\alpha}_{\bar{1}}^+\alpha_{\bar{2}}^-\bar{\alpha}_{\bar{3}}^+\alpha_{\bar{4}}^-)\\
&+\frac{2}{(\hat{1}+\hat{4})^2}\bar{\alpha}_{\bar{1}}^+\alpha_{\bar{2}}^+\bar{\alpha}_{\bar{3}}^-\alpha_{\bar{4}}^- + \frac{1}{N_{\mathrm{C}}}\frac{1}{(\hat{1}+\hat{2})^2}\bar{\alpha}_{\bar{1}}^-\alpha_{\bar{2}}^+\bar{\alpha}_{\bar{3}}^+\alpha_{\bar{4}}^-\Bigg\}.
\end{aligned} \tag{3.7}$$

We substitute for $\bar{\mathcal{A}}$ in (3.4) using the leading order terms in (2.59) and for the fermions in (3.7). Summing and symmetrising over the momentum labels as much as possible leads to the following terms in the MHV lagrangian:

$$\begin{aligned}
\frac{g^4}{8}\int_{1234}\Bigg\{&\frac{1}{2}\left(\frac{(2\,4)^2}{\hat{2}\hat{4}(1\,4)(3\,2)}+\frac{1}{N_{\mathrm{C}}}\frac{(2\,4)^2}{\hat{2}\hat{4}(1\,2)(3\,4)}\right)\bar{\xi}_{\bar{1}}^{+\bar{\imath}_1}\xi_{\bar{2}\;i_2}^-\bar{\xi}_{\bar{3}}^{+\bar{\imath}_3}\xi_{\bar{4}\;i_4}^-\\
&+\frac{1}{2}\left(\frac{(1\,3)^2}{\hat{1}\hat{3}(1\,4)(3\,2)}+\frac{1}{N_{\mathrm{C}}}\frac{(1\,3)^2}{\hat{1}\hat{3}(1\,2)(3\,4)}\right)\bar{\xi}_{\bar{1}}^{-\bar{\imath}_1}\xi_{\bar{2}\;i_2}^+\bar{\xi}_{\bar{3}}^{-\bar{\imath}_3}\xi_{\bar{4}\;i_4}^+\\
&-\frac{(3\,4)^2}{\hat{3}\hat{4}(1\,4)(3\,2)}\bar{\xi}_{\bar{1}}^{+\bar{\imath}_1}\xi_{\bar{2}\;i_2}^+\bar{\xi}_{\bar{3}}^{-\bar{\imath}_3}\xi_{\bar{4}\;i_4}^- - \frac{1}{N_{\mathrm{C}}}\frac{(2\,3)^2}{\hat{2}\hat{3}(1\,2)(3\,4)}\bar{\xi}_{\bar{1}}^{+\bar{\imath}_1}\xi_{\bar{2}\;i_2}^-\bar{\xi}_{\bar{3}}^{-\bar{\imath}_3}\xi_{\bar{4}\;i_4}^+\Bigg\}\delta_{i_1}^{\bar{\imath}_2}\delta_{i_3}^{\bar{\imath}_4}.
\end{aligned} \tag{3.8}$$

[4]This is equivalent to subtracting the 'fictitious photon', as found in the literature that works with a $\mathrm{U}(N_{\mathrm{C}})$ gauge symmetry instead

Writing it out this way makes immediate contact with the second term of (2.37), and we can read off the vertices immediately:

$$V_{\mathrm{F}}^{2,+-+-}(1234) = \frac{g^4}{8}\left(\frac{(2\,4)^2}{\hat{2}\hat{4}(1\,4)(3\,2)} + \frac{1}{N_{\mathrm{C}}}\frac{(2\,4)^2}{\hat{2}\hat{4}(1\,2)(3\,4)}\right), \tag{3.9}$$

$$V_{\mathrm{F}}^{2,-+-+}(1234) = \frac{g^4}{8}\left(\frac{(1\,3)^2}{\hat{1}\hat{3}(1\,4)(3\,2)} + \frac{1}{N_{\mathrm{C}}}\frac{(1\,3)^2}{\hat{1}\hat{3}(1\,2)(3\,4)}\right), \tag{3.10}$$

$$V_{\mathrm{F}}^{2,++--}(1234) = -\frac{g^4}{8}\frac{(3\,4)^2}{\hat{3}\hat{4}(1\,4)(3\,2)}, \tag{3.11}$$

$$V_{\mathrm{F}}^{2,+--+}(1234) = -\frac{g^4}{8}\frac{1}{N_{\mathrm{C}}}\frac{(2\,3)^2}{\hat{2}\hat{3}(1\,2)(3\,4)}, \tag{3.12}$$

which are readily seen to concur with (2.87) upon using (2.75)–(2.78) and (2.6).

3.3 Two quarks and three gluons

Finally, we consider the vertex $V_{\mathrm{F}}^{3,+-}(12345)$. In the MHV lagrangian, this is the coefficient of the term in $\bar{\xi}^+_{\bar{1}}\mathcal{B}_{\bar{2}}\bar{\mathcal{B}}_{\bar{3}}\mathcal{B}_{\bar{4}}\xi^-_{\bar{5}}$ and it receives contributions from L^{--+}, L^{--++}, $L^{\bar{\psi}-\psi}$ and $L^{\bar{\psi}+-\psi}$, so we will write it as

$$\int_{12345}(W^{--+} + W^{--++} + W^{\bar{\psi}-\psi} + W^{\bar{\psi}+-\psi})\bar{\xi}^+_{\bar{1}}\mathcal{B}_{\bar{2}}\bar{\mathcal{B}}_{\bar{3}}\mathcal{B}_{\bar{4}}\xi^-_{\bar{5}}\,. \tag{3.13}$$

Let us consider each of the Ws in turn. First, we observe from (3.1) and the structures of (2.57) and (2.59) that L^{--+} yields four terms with the structure of (3.13) coming from the different possible choices substitution for $\mathcal{A}$. We carry this out and carefully relabel the momenta while accounting for the anticommuting nature of the fermions to obtain

$$\begin{aligned} W^{--+} = \frac{ig^2}{\sqrt{8}}\Bigg\{ & \frac{\hat{2}\hat{3}^2\hat{5}(2,3+4)}{(\hat{1}+\hat{5})^3(\hat{3}+\hat{4})^3}\Upsilon(-,3,4)\Upsilon(-,5,1) \\ & + \frac{\hat{3}^2\hat{4}\hat{5}(2+3,4)}{(\hat{1}+\hat{5})^3(\hat{2}+\hat{3})^3}\Upsilon(-,2,3)\Upsilon(-,5,1) \\ & - \frac{\hat{4}\hat{5}(3\,4)}{\hat{3}(\hat{3}+\hat{4})^3}\Upsilon(-,5,1,2) + \frac{\hat{2}\hat{5}(2\,3)}{\hat{3}(\hat{2}+\hat{3})^3}\Upsilon(-,4,5,1)\Bigg\}. \end{aligned} \tag{3.14}$$

We remind the reader here that in the argument lists of Υ, K, *etc.*, $-$ is a placeholder whose value should be taken to be the negative of the sum of the other momenta passed to that coefficient.

Next, L^{--++} has terms of the form $\mathrm{tr}(\bar{\mathcal{A}}\mathcal{A}\bar{\mathcal{A}}\mathcal{A})$ and $\mathrm{tr}(\bar{\mathcal{A}}\bar{\mathcal{A}}\mathcal{A}\mathcal{A})$, of which only the former contribute terms in (3.13). In momentum space, this is

$$\mathrm{tr}\int_{1234}\left\{\frac{\hat{2}\hat{3}}{(\hat{3}+\hat{4})^2} + \frac{\hat{3}\hat{4}}{(\hat{2}+\hat{3})^2}\right\}\bar{\mathcal{A}}_{\bar{1}}\mathcal{A}_{\bar{2}}\bar{\mathcal{A}}_{\bar{3}}\mathcal{A}_{\bar{4}}. \tag{3.15}$$

Substituting for each $\bar{\mathcal{A}}$ in turn using the lowest-order terms in (2.59) gives

$$
\begin{aligned}
W^{--++} = \frac{g^2}{\sqrt{8}} \frac{\hat{5}}{(\hat{1}+\hat{5})^2} \Bigg\{ & \left(\frac{\hat{2}\hat{3}}{(\hat{3}+\hat{4})^2} + \frac{\hat{3}\hat{4}}{(\hat{2}+\hat{3})^2} \right) \Upsilon(-,5,1) \\
& + \left(\frac{\hat{4}(\hat{1}+\hat{5})}{(\hat{3}+\hat{4})^2} + \frac{\hat{2}(\hat{1}+\hat{5})}{(\hat{2}+\hat{3})^2} \right) \Upsilon(-,5,1) \Bigg\}. \qquad (3.16)
\end{aligned}
$$

$L^{\bar{\psi}-\psi}$ contributes four terms in the structure of (3.13), owing to the fact that we will now also see terms from the fermion series expansions (2.33) and (2.32) when these are substituted into (3.4). Upon re-arrangement and re-labelling of the momenta, we arrive at the contribution

$$
\begin{aligned}
W^{\bar{\psi}-\psi} = -\frac{ig^2}{\sqrt{8}} \Bigg\{ & \left(\frac{\tilde{4}+\tilde{5}}{\hat{4}+\hat{5}} - \frac{\tilde{3}}{\hat{3}} \right) \frac{\hat{5}}{\hat{4}+\hat{5}} \Upsilon(-,1,2)\Upsilon(-,4,5) \\
& + \left(\frac{\hat{3}}{\hat{3}+\hat{4}} \right)^2 \left(\frac{\tilde{5}}{\hat{5}} - \frac{\tilde{3}+\tilde{4}}{\hat{3}+\hat{4}} \right) \Upsilon(-,1,2)\Upsilon(-,3,4) \\
& + \left(\frac{\hat{3}}{\hat{2}+\hat{3}+\hat{4}} \right)^2 \left(\frac{\tilde{5}}{\hat{5}} - \frac{\tilde{1}+\tilde{5}}{\hat{1}+\hat{5}} \right) \Upsilon(-,2,3,4) \\
& + \left(\frac{\hat{3}}{\hat{2}+\hat{3}} \right)^2 \left(\frac{\tilde{4}+\tilde{5}}{\hat{4}+\hat{5}} - \frac{\tilde{2}+\tilde{3}}{\hat{2}+\hat{3}} \right) \frac{\hat{5}}{\hat{4}+\hat{5}} \Upsilon(-,2,3)\Upsilon(-,4,5) \Bigg\}. \qquad (3.17)
\end{aligned}
$$

Finally, $L^{\bar{\psi}+-\psi}$, which has momentum-space representation

$$
\begin{aligned}
L^{\bar{\psi}+-\psi} = -\frac{g^2}{\sqrt{8}} \int_{1234} \Bigg\{ & \left(\frac{1}{\hat{3}+\hat{4}} + \frac{\hat{2}-\hat{3}}{(\hat{2}+\hat{3})^2} \right) \bar{\alpha}^+_{\bar{1}} \bar{\mathcal{A}}_{\bar{2}} \mathcal{A}_{\bar{3}} \alpha^-_{\bar{4}} \\
& + \frac{\hat{2}-\hat{3}}{(\hat{2}+\hat{3})^2} \bar{\alpha}^+_{\bar{1}} \mathcal{A}_{\bar{2}} \bar{\mathcal{A}}_{\bar{3}} \alpha^-_{\bar{4}} + \text{l.h. pieces} \Bigg\}, \qquad (3.18)
\end{aligned}
$$

contributes four terms to (3.13) from substitutions for $\bar{\mathcal{A}}$ and the fermions:

$$
\begin{aligned}
W^{\bar{\psi}+-\psi} = -\frac{g^2}{\sqrt{8}} \Bigg\{ & \left(\frac{1}{\hat{4}+\hat{5}} + \frac{\hat{3}-\hat{4}}{(\hat{3}+\hat{4})^2} \right) \Upsilon(-,1,2) + \frac{\hat{2}-\hat{3}}{(\hat{2}+\hat{3})^2} \frac{\hat{5}}{\hat{4}+\hat{5}} \Upsilon(-,4,5) \\
& + \left(\frac{\hat{3}}{\hat{3}+\hat{4}} \right)^2 \frac{\hat{2}-\hat{3}-\hat{4}}{(\hat{2}+\hat{3}+\hat{4})^2} \Upsilon(-,3,4) \\
& + \left(\frac{\hat{3}}{\hat{2}+\hat{3}} \right)^2 \left(\frac{1}{\hat{4}+\hat{5}} + \frac{\hat{2}+\hat{3}-\hat{4}}{(\hat{2}+\hat{3}+\hat{4})^2} \right) \Upsilon(-,2,3) \Bigg\}. \qquad (3.19)
\end{aligned}
$$

$V_{\mathrm{F}}^{3,+-}(12345)$ is obtained from the sum of the coefficients in (3.14), (3.16), (3.17) and (3.19) and is

$$
V_{\mathrm{F}}^{3,+-}(12345) = \frac{ig^2}{\sqrt{8}} \frac{\hat{2}\hat{4}}{\hat{3}\hat{5}} \frac{(1\,3)(3\,5)^3}{(1\,2)(2\,3)(3\,4)(4\,5)(5\,1)}, \qquad (3.20)
$$

which satisfies (2.84), so the vertex is proportional to the MHV amplitude.

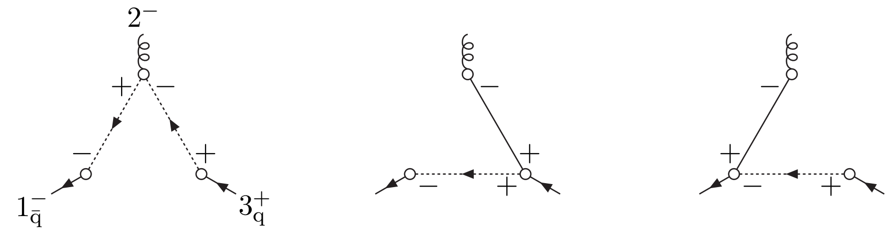


Figure 2: Contributions to the tree-level $A(1_{\mathrm{q}}^+ 2^+ 3_{\bar{\mathrm{q}}}^-)$ amplitude, before applying LSZ reduction. All momenta are directed out of the diagrams, arrows indicate colour flow.

3.4 On completion vertices and missing amplitudes

We showed in [25] that the S-matrix receives contributions from more than just the vertex content of a MHV lagrangian. The complete treatment of the S-matrix via the LSZ reduction revealed completion vertices that arose from the terms in the transformation itself. We found that they are required to obtain amplitudes that can only be formed by using vertices eliminated by the field transformation. At tree-level, these can be non-vanishing on shell in $(2,2)$ signature or for complex momenta and they are needed for the complete recovery of the full off-shell theory at tree-level. At loop level, they are needed in general to recover the full physical on-shell amplitudes. We demonstrated this by calculating the $A(1^- 2^+ 3^+)$ tree-level amplitude and showing that $A(1^+ 2^+ 3^+ 4^+)$ at one-loop matches the known result, as obtained using light-cone Yang-Mills theory.

The situation when one adds quarks is no different: quark-gluon amplitudes whose construction requires erstwhile $\overline{\text{MHV}}$-like vertices are recovered through completion vertices found in (2.39), (2.32), (2.33) and (2.59). As an example of this, let us study the partial amplitude $A(1_{\mathrm{q}}^+ 2^+ 3_{\bar{\mathrm{q}}}^-)$. It is given by

$$A(1_{\mathrm{q}}^+ 2^+ 3_{\bar{\mathrm{q}}}^-) = ig \frac{[2\,1]^2}{[3\,1]}. \tag{3.21}$$

The LSZ reduction gives this amplitude as

$$A(1_{\mathrm{q}}^+ 2^+ 3_{\bar{\mathrm{q}}}^-) = \lim_{p_1^2, p_2^2, p_3^2 \to 0} i p_2^2 \times \frac{-i p_1^2}{2^{1/4}\sqrt{\hat{1}}} \times \frac{-i p_3^2}{2^{1/4}\sqrt{\hat{3}}} \times \langle \alpha_1^- \, \bar{A}_2 \, \bar{\alpha}_3^+ \rangle. \tag{3.22}$$

Here, the first factor contributes the gluon polarisation and inverse propagator; and the second and third factors are (2.82) for the quarks. The correlation function may be computed by substituting for each of the fields involved with their next-to-leading-order expressions from (2.32), (2.33) and (2.59), or equivalently by evaluating the sum of the three diagrams of figure 2, constructed from the vertices of figure 1. Thus, (3.22) becomes

$$
\begin{aligned}
A(1_{\rm q}^+2^+3_{\bar{\rm q}}^-) &= -\frac{i}{\sqrt{2}\,\hat{1}\hat{3}}p_1^2p_2^2p_3^2\left\{\frac{\hat{1}}{p_1^2}\frac{\hat{3}}{p_3^2}\frac{ig}{\hat{2}^2}\hat{3}\Upsilon(231)+\frac{\hat{1}}{p_1^2}\frac{1}{p_2^2}\,ig\,\Upsilon(312)+\frac{1}{p_2^2}\frac{\hat{3}}{p_3^2}\,ig\,\frac{\hat{3}}{\hat{1}}\Upsilon(123)\right\}\\
&= \frac{ig}{\sqrt{2}}\sqrt{\hat{1}\hat{3}}\frac{\hat{3}}{(1\,2)}\left(\frac{p_1^2}{\hat{1}}+\frac{p_2^2}{\hat{2}}+\frac{p_3^2}{\hat{3}}\right)\\
&= ig\sqrt{2}\frac{1}{\hat{2}}\sqrt{\frac{\hat{3}}{\hat{1}}}\,\{3\,1\}.
\end{aligned}
\tag{3.23}
$$

where we have used (2.49), (2.52), (2.53), (2.67) and (2.69), and restored the canonical normalisation of A using (2.15). This expression may be shown to equal (3.21) using (2.6). We obtained the final line using the result [25]

$$
\sum_j \frac{(p\,j)\cdot\{j\,q\}}{\hat{j}} = \frac{\hat{p}\,\hat{q}}{2}\sum_j \frac{p_j^2}{\hat{j}} \tag{3.24}
$$

which holds for any set $\{p_i\}$ of momenta that sums to zero.

4. Massive quarks

In this section, we analyse the MHV QCD lagrangian in the case where the quarks have masses. For simplicity, we will consider the case of a single flavour of quark with mass m; the flavour-conserving nature of QCD makes the generalisation of the procedure to multiple flavours straightforward.

To begin, we add a Dirac mass term to the original lagrangian, so

$$
\bar{\psi}\, i\not{D}\,\psi \to \bar{\psi}(i\not{D}-m)\psi \tag{4.1}
$$

in (2.13). If we repeat the analysis of section 2.3, we again arrive at a light-cone action given by (2.17) with the following additional terms in the parentheses:

$$
\begin{aligned}
L_m^{\bar{\psi}\psi} &= \frac{im^2g^2}{4\sqrt{2}}\int_\Sigma d^3\mathbf{x}(\bar{\alpha}^-\hat{\partial}^{-1}\alpha^+ + \bar{\alpha}^+\hat{\partial}^{-1}\alpha^-), && (4.2)\\
L^{\bar{\psi}^+-\psi^+} &= -\frac{mg^2}{4}\int_\Sigma d^3\mathbf{x}\,\bar{\alpha}^+[\hat{\partial}^{-1},\bar{\mathcal{A}}]\alpha^+, && (4.3)\\
L^{\bar{\psi}^-+\psi^-} &= \frac{mg^2}{4}\int_\Sigma d^3\mathbf{x}\,\bar{\alpha}^-[\hat{\partial}^{-1},\mathcal{A}]\alpha^-. && (4.4)
\end{aligned}
$$

Note now that since $m \neq 0$, chirality and helicity become mixed; therefore the signs in the superscripts of the quark fields no longer denote the corresponding out-going particles' helicities. With the term (4.2), the propagator for the light-cone gauge action's quarks becomes

$$
\frac{i\sqrt{2}\,\hat{p}}{p^2-m^2}. \tag{4.5}
$$

We note that the quark propagator conserves the chirality even in the massive case, and that the only chirality violation comes from the vertices in (4.3) and (4.4).

The polarisation spinors for massive states have at least three non-zero components, and so couple to non-dynamical quark components when applied in the LSZ reduction. To understand how we should deal with this, let us study the gauge-fixing process with the source terms instated. They contribute the following term to the action:

$$S_{\rm src} = \int d^4x \left\{ \frac{i\sqrt{2}}{g} \operatorname{tr} J^\mu \mathcal{A}_\mu + \bar{\kappa}\psi + \bar{\psi}\kappa \right\}, \tag{4.6}$$

where the fermionic sources are

$$\kappa = (\kappa_1^+, \kappa_2^+, \kappa_2^-, \kappa_1^-)^{\rm T} \quad \text{and} \quad \bar{\kappa} = (\bar{\kappa}_2^+, \bar{\kappa}_1^+, \bar{\kappa}_1^-, \bar{\kappa}_2^-). \tag{4.7}$$

Fixing $\hat{\mathcal{A}} = 0$ and integrating out $\check{\mathcal{A}}$ and the non-dynamical fermions leaves

$$\begin{aligned} S'_{\rm src} = \int d^4x \Bigg\{ & \bar{\kappa}_1^+ \hat{\partial}^{-1} \left(\bar{\mathcal{D}}\alpha^+ - \frac{im}{\sqrt{2}}\alpha^- \right) - \bar{\kappa}_1^- \hat{\partial}^{-1} \left(\mathcal{D}\alpha^- + \frac{im}{\sqrt{2}}\alpha^+ \right) \\ & + \hat{\partial}^{-1} \left(\mathcal{D}\bar{\alpha}^- + \frac{im}{\sqrt{2}}\bar{\alpha}^+ \right) \kappa_1^- - \hat{\partial}^{-1} \left(\bar{\mathcal{D}}\bar{\alpha}^+ - \frac{im}{\sqrt{2}}\bar{\alpha}^- \right) \kappa_1^+ \\ & + \frac{i}{\sqrt{2}} (\bar{\kappa}_1^+ \hat{\partial}^{-1} \kappa_1^- + \bar{\kappa}_1^- \hat{\partial}^{-1} \kappa_1^+) \\ & + \bar{\kappa}_2^+ \alpha^+ + \bar{\kappa}_2^- \alpha^- + \bar{\alpha}^+ \kappa_2^+ + \bar{\alpha}^- \kappa_2^- \\ & - \frac{i\sqrt{2}}{g} \operatorname{tr}[\bar{J}\mathcal{A} + J\bar{\mathcal{A}}] + \hat{J} \text{ terms} \Bigg\}. \end{aligned} \tag{4.8}$$

The important thing to notice here, even before we have applied any field transformation, is what has happened to the sources $\kappa_1^\pm$ and $\bar{\kappa}_1^\pm$. These were previously coupled to non-dynamical quark field components $\beta^\pm$ and $\bar{\beta}^\pm$, but from (4.8) we see that functional derivatives of the partition function with respect to these sources bring down insertions of a linear combination of the dynamical quark degrees of freedom, and a completion vertex-like product of a gauge and quark field. At the tree-level, the latter should be annihilated for generic momenta as we take the on-shell limit in the LSZ theorem. In contrast, we would initially expect the former to contribute. However, as we shall see in section 4.1, we can choose our polarisation spinors so that the non-dynamical components are once again decoupled.

Let us turn to the field transformation for the massive scenario. First, suppose we hold on to much of the machinery of section 2: specifically, we retain the definition of the transformation in (2.27), excluding the mass terms. Since the definitions of the canonical momenta (2.28) do not change, the calculation carries through as before, and we then substitute the expressions we have obtained for the old fields into (4.2)–(4.4); the resulting lagrangian will be the massless MHV lagrangian with additional terms proportional to m^2 and m.[5] This is the most straightforward way of extending to the massive quark case from the massless one (we consider another possible transformation just before section 4.1). Notice that the term of (4.3) would appear to be of the 'wrong' form for a MHV lagrangian,

[5]This method is also adopted in [31] and [32] in their discussion of massive scalars.

but we must keep the following in mind: it is of course no longer correct any more to talk of the resulting terms as being of a maximally *helicity*-violating form since the quark fields in each term only have definite chirality, not helicity, when masses are present. Nevertheless, in each term the number of negative helicity gauge fields is still constrained to be not more than two, and any given term never contains any more than four quark fields, so for future work on this topic it may be useful to classify vertices according to their chirality content.

With these choices in place, the massive MHV lagrangian consists of: the massless kinetic terms for the gluons and quarks (the mass terms for the latter are discussed below); the massless QCD interaction terms (2.35) and (2.37); and now a new set of massive terms arising from the substitution of the field transformation into (4.2)–(4.4). These new massive pieces provide the MHV lagrangian with terms of the form

$$L^{+-}_{\mathrm{F}m} = \sum_{n=2}^{\infty}\int_{1\cdots n}\left\{V^{-+}_{\mathrm{F}m}(1\cdots n)\bar{\xi}^-_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^+_{\bar{n}} + V^{+-}_{\mathrm{F}m}(1\cdots n)\bar{\xi}^+_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^-_{\bar{n}}\right\}, \quad (4.9)$$

$$\begin{aligned} L^{+-+}_{\mathrm{F}m} = &\sum_{n=3}^{\infty}\sum_{s=2}^{n-1}\int_{1\cdots n}V^{s,+-+}_{\mathrm{F}m}(1\cdots n)\bar{\xi}^+_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\bar{\mathcal{B}}_{\bar{s}}\cdots\mathcal{B}_{\overline{n-1}}\xi^+_{\bar{n}} \\ &+\sum_{n=4}^{\infty}\sum_{s=2}^{n-2}\int_{1\cdots n}\Big\{V^{s,++-+}_{\mathrm{F}m}(1\cdots n)\bar{\xi}^+_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{s-1}}\xi^+_{\bar{s}}\,\bar{\xi}^-_{\overline{s+1}}\mathcal{B}_{\overline{s+2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^+_{\bar{n}} \\ &\qquad + V^{s,+-++}_{\mathrm{F}m}(1\cdots n)\bar{\xi}^+_{\bar{1}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{s-1}}\xi^-_{\bar{s}}\,\bar{\xi}^+_{\overline{s+1}}\mathcal{B}_{\overline{s+2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^+_{\bar{n}}\Big\}, \end{aligned} \quad (4.10)$$

$$L^{-+-}_{\mathrm{F}m} = \sum_{n=3}^{\infty}\int_{1\cdots n}V^{-+-}_{\mathrm{F}m}(1\cdots n)\bar{\xi}^-_{\bar{1}}\mathcal{B}_{\bar{2}}\mathcal{B}_{\bar{3}}\cdots\mathcal{B}_{\overline{n-1}}\xi^-_{\bar{n}}\,. \quad (4.11)$$

(The momentum-conserving δ function has been absorbed into the integration measure as in (2.35) and (2.37).) It is straightforward though tedious to calculate these vertices by substituting the series for $\mathcal{A}$, $\bar{\mathcal{A}}$, α, $\bar{\alpha}$ in terms of $\mathcal{B}$, $\bar{\mathcal{B}}$, ξ, $\bar{\xi}$ into (4.2)–(4.4):

$$V^{-+}_{\mathrm{F}m} = \frac{(-i)^{n-2}m^2g^2}{4\sqrt{2}}\frac{\hat{1}\hat{2}\cdots\widehat{n-1}\,(1\,n)}{\hat{1}\hat{n}(1\,2)(2\,3)\cdots(n-1,n)} \quad \text{for } n\geq 2, \quad (4.12)$$

$$V^{+-}_{\mathrm{F}m} = -\frac{(-i)^{n-2}m^2g^2}{4\sqrt{2}}\frac{\hat{2}\cdots\widehat{n-1}\,\hat{n}(1\,n)}{\hat{1}\hat{n}(1\,2)(2\,3)\cdots(n-1,n)} \quad \text{for } n\geq 2, \quad (4.13)$$

$$V^{s,+-+}_{\mathrm{F}m} = \frac{(-i)^{n-2}mg^2}{4}\frac{\hat{2}\hat{3}\cdots\widehat{n-1}\,(1\,s)(s\,n)}{\hat{1}\hat{n}(1\,2)(2\,3)\cdots(n-1,n)} \quad \text{for } n\geq 3, \quad (4.14)$$

$$V^{-+-}_{\mathrm{F}m} = -\frac{(-i)^{n-2}mg^2}{4}\frac{\hat{2}\hat{3}\cdots\widehat{n-1}\,(1\,n)^2}{\hat{1}\hat{n}(1\,2)(2\,3)\cdots(n-1,n)} \quad \text{for } n\geq 3, \quad (4.15)$$

$$V^{s,++-+}_{\mathrm{F}m} = \frac{(-i)^{n-2}mg^4}{8\sqrt{2}}\frac{\hat{2}\hat{3}\cdots\widehat{n-1}\,(1\,s)}{\hat{1}\hat{s}(1\,2)(2\,3)\cdots(n-1,n)}\left(\frac{(s+1,n)}{\hat{n}}+\frac{1}{N_C}\frac{(s,s+1)}{\hat{s}}\right) \quad (4.16)$$

for $n\geq 4$,

$$V^{s,+-++}_{\mathrm{F}m} = -\frac{(-i)^{n-2}mg^4}{8\sqrt{2}}\frac{\hat{2}\hat{3}\cdots\widehat{n-1}\,(s+1,n)}{\widehat{s+1}\hat{n}(1\,2)(2\,3)\cdots(n-1,n)}\left(\frac{(1\,s)}{\hat{1}}+\frac{1}{N_C}\frac{(s,s+1)}{\widehat{s+1}}\right) \quad (4.17)$$

for $n\geq 4$.

(Note above that we have dropped the argument list $(1\cdots n)$ from the left-hand sides.) Similarly to the amplitudes, the vertices have the relations:

$$\begin{aligned} V_{\mathrm{F}m}^{+-}(1\cdots n) &= (-1)^{n+1} V_{\mathrm{F}m}^{-+}(n\cdots 1), \\ V_{\mathrm{F}m}^{s,+-+}(1\cdots n) &= (-1)^{n+1} V_{\mathrm{F}m}^{n-s,+-+}(n\cdots 1), \\ V_{\mathrm{F}m}^{s,++-+}(1\cdots n) &= (-1)^{n} \quad V_{\mathrm{F}m}^{n-s,+-++}(n\cdots 1), \\ V_{\mathrm{F}m}^{-+-}(1\cdots n) &= (-1)^{n+1} V_{\mathrm{F}m}^{-+-}(n\cdots 1). \end{aligned}$$

Equations (4.12) and (4.13) are proportional to the CSW vertices for massive scalars as derived in [31], which may imply a supersymmetric relation between these two kinds of vertices.

Since to leading order α is just ξ, the leading order terms of the expansion proportional to m^2 in the MHV lagrangian (i.e. the $n = 2$ terms in (4.9)) are obtained by replacing α with ξ in (4.2). Also, the series expansions are also the expansion with respect to g, a factor of which is hidden in each $\mathcal{B}$. If we consider the canonically normalised form of the action, including the $4/g^2$ factor from (2.17), we see that the terms in (4.9) form a series in g^{n-2} for $n \geq 2$. Thus we merge the $n = 2$ terms in (4.9) into the kinetic part on which the perturbative expansion is based, and the higher order expansion terms into the interaction part. In this way, the propagator for the new fermion fields is just the same as that of the old ones:

$$\langle \xi^{\pm} \bar{\xi}^{\mp} \rangle = \frac{i\sqrt{2}\,\hat{p}}{p^2 - m^2}\,, \tag{4.18}$$

which is just the corresponding component of the ordinary propagator $i(\not{p} + m)/(p^2 - m^2)$. Similarly, we see that the terms that (4.3) and (4.4) supply to the MHV lagrangian, which are also proportional to m, generate a perturbation series in g starting at $\mathcal{O}(g)$. In section 4.4, as a sample calculation, we will check the three-particle amplitudes obtained from the MHV lagrangian against those from ordinary massive light-cone gauge QCD.

Parenthetically, we note that one might choose another way of generalising to massive quarks, specifically including the massive terms in the transformation equation according to

$$\begin{aligned} &L^{-+}[\mathcal{A}, \bar{\mathcal{A}}] + L^{-++}[\mathcal{A}, \bar{\mathcal{A}}] + L^{\bar{\psi}\psi}[\alpha^{\pm}, \bar{\alpha}^{\pm}] + L^{\bar{\psi}+\psi}[\mathcal{A}, \alpha^{\pm}, \bar{\alpha}^{\pm}] + L_m^{\bar{\psi}\psi}[\alpha^{\pm}, \bar{\alpha}^{\pm}] \\ &+ L^{\bar{\psi}^{+}-\psi^{+}}[\alpha^{+}, \bar{\mathcal{A}}] = L^{-+}[\mathcal{B}, \bar{\mathcal{B}}] + L^{\bar{\psi}\psi}[\xi^{\pm}, \bar{\xi}^{\pm}] + L_m^{\bar{\psi}\psi}[\xi^{\pm}, \bar{\xi}^{\pm}]. \end{aligned} \tag{4.19}$$

The difficulty with such a transformation is that there is no reason to expect a simple closed form for the coefficients of the expansion in the new fields any more (something similar was seen in ref. [25], when considering the pure Yang-Mills MHV lagrangian outside four dimensions). On the other hand, one may be able to derive closed forms for the coefficients of the expansion in the new fields if one also simultaneously expands in an infinite series of higher powers in m. This alternative approach is clearly more complicated than the one we pursue here, in which the expansion coefficients have a simple elegant closed form independent of the quark mass, and the lagrangian only has terms with powers of the quark mass not exceeding two.

4.1 Massive polarisation spinors

The Dirac equations in momentum space are

$$(\not{p} - m)u^\sigma(\mathbf{p}) = 0 \quad \text{and} \quad (-\not{p} - m)v^\sigma(\mathbf{p}) = 0\,, \tag{4.20}$$

where u and v are the positive- and negative-energy solutions, respectively. Now, for the purposes of the LSZ reduction, we can choose our polarisation spinors to satisfy these equations, but instead with a r.h.s. that vanishes only on shell. Furthermore, we can choose this r.h.s. to have components that result in $\beta^\pm$ and $\bar{\beta}^\pm$ being decoupled from the reduction as in equations (2.81) and (2.82) in massless cases, i.e. we search for off-shell solutions u^σ and v^σ that satisfy

$$(\not{p} - m)u^\sigma(p) = \begin{pmatrix} 0 \\ * \\ * \\ 0 \end{pmatrix} (p^2 - m^2), \quad (\not{p} + m)v^\sigma(p) = \begin{pmatrix} 0 \\ * \\ * \\ 0 \end{pmatrix} (p^2 - m^2). \tag{4.21}$$

where the asterisks denote unspecified quantities.

Let us concentrate on the positive-energy spinor u and postulate solutions of the form

$$u^\pm = \begin{pmatrix} a^\pm \eta_1^\pm \\ b^\pm \eta_2^\pm \end{pmatrix} \tag{4.22}$$

where

$$\eta_1^+ = \eta^+ \qquad \text{and} \qquad \eta_2^+ = \eta^+ + \rho^+ \begin{pmatrix} p^2 - m^2 \\ 0 \end{pmatrix}, \tag{4.23}$$

$$\eta_2^- = \eta^- \qquad \text{and} \qquad \eta_1^- = \eta^- + \rho^- \begin{pmatrix} 0 \\ p^2 - m^2 \end{pmatrix}. \tag{4.24}$$

Here, the $\eta^\pm$ are the eigenspinors of the helicity operator and solve the equation

$$\frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{|\mathbf{p}|} \eta^\lambda = \lambda \eta^\lambda. \tag{4.25}$$

This has two solutions for $\lambda = \pm$,

$$\eta^+ = -\sqrt{2}\Big(p, p^- - \hat{p}\Big)^{\mathrm{T}}, \quad \text{and} \quad \eta^- = -\sqrt{2}\Big(p^- - \hat{p}, -\bar{p}\Big)^{\mathrm{T}}, \tag{4.26}$$

in which we have defined

$$p^\pm := \frac{1}{\sqrt{2}}(p^t \pm |\mathbf{p}|) \tag{4.27}$$

in order to tidy up notations in the following equations. In the massless limit, $\eta^\pm$ recovers φ, $\bar{\omega}$ in (2.9) and (2.10) up to a normalization.

By solving (4.21), and using the normalization $u^\dagger u = 2p^t$, the $u^\pm$ are

$$u^+(p) = \nu \begin{pmatrix} -\sqrt{2}mp \\ -\sqrt{2}m(p^- - \hat{p}) \\ -p\big[m^2 - 2p^+(p^- - \hat{p})\big]/\hat{p} \\ -2p^+(p^- - \hat{p}) \end{pmatrix}, \qquad (4.28)$$

and

$$u^-(p) = \nu \begin{pmatrix} -2p^+(p^- - \hat{p}) \\ \bar{p}\big[m^2 - 2p^+(p^- - \hat{p})\big]/\hat{p} \\ -\sqrt{2}m(p^- - \hat{p}) \\ \sqrt{2}m\bar{p} \end{pmatrix}, \qquad (4.29)$$

where

$$\nu = \frac{1}{2\sqrt{|\mathbf{p}|p^+(\hat{p} - p^-)}}\,. \qquad (4.30)$$

The normalisation ν is just an overall coefficient and does not affect the off-shell condition of (4.21). We have therefore set $p^2 = m^2$ within it to simplify the result. It is also easy to verify that in the massless limit the solution reduces to the previous massless $u^\pm$ using the limiting behavior:

$$p^+ \to \sqrt{2}p^t\,, \quad p^- \to \frac{m^2}{2\sqrt{2}p^t}\,, \quad \hat{p} - p^+ \to -\check{p}\,, \quad \hat{p} - p^- \to \hat{p}\,, \quad \nu \to \frac{1}{2^{5/4}p^t\sqrt{\hat{p}}}\,. \qquad (4.31)$$

For the antiquark, we replace $m \to -m$ and flip the helicity in above equations, i.e. $v^\pm = u^\mp|_{m\to -m}$. If we define:

$$\bar{\phi}^{\dot{\alpha}} = \nu \begin{pmatrix} -\sqrt{2}mp \\ -\sqrt{2}m(p^- - \hat{p}) \end{pmatrix} \quad \text{and} \quad \chi_\alpha = \nu \begin{pmatrix} -p\big[m^2 - 2p^+(p^- - \hat{p})\big]/\hat{p} \\ -2p^+(p^- - \hat{p}) \end{pmatrix}, \qquad (4.32)$$

$u^\pm$ and $v^\pm$ can be represented in a more compact form, where we have used $\epsilon^{\dot{\alpha}\dot{\beta}}$ and $\epsilon_{\alpha\beta}$ to raise and lower spinor indices:

$$u^+ = \begin{pmatrix} \bar{\phi}^{\dot{\alpha}} \\ \chi_\alpha \end{pmatrix}, \quad u^- = \begin{pmatrix} \bar{\chi}^{\dot{\alpha}} \\ -\phi_\alpha \end{pmatrix}, \quad v^+ = \begin{pmatrix} \bar{\chi}^{\dot{\alpha}} \\ \phi_\alpha \end{pmatrix}, \quad v^- = \begin{pmatrix} -\bar{\phi}^{\dot{\alpha}} \\ \chi_\alpha \end{pmatrix}. \qquad (4.33)$$

Explicit calculation shows that these solutions are orthogonal for off-shell p, i.e. $\bar{u}^+(p)\gamma^0 u^-(p) = \bar{u}^+(p)u^-(p) = \bar{u}(p)v(p) = 0$.

Consequently, (4.21) becomes

$$(\not{p} - m)u^+(p) = \frac{\nu}{\hat{p}} \begin{pmatrix} 0 \\ -\sqrt{2}p^+(p^- - \hat{p}) \\ -mp \\ 0 \end{pmatrix} (p^2 - m^2)\,, \qquad (4.34)$$

$$(\not{p} - m)u^-(p) = \frac{\nu}{\hat{p}} \begin{pmatrix} 0 \\ m\bar{p} \\ -\sqrt{2}p^+(p^- - \hat{p}) \\ 0 \end{pmatrix} (p^2 - m^2)\,, \qquad (4.35)$$

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$$(\not{p}+m)v^{+}(p)=\frac{\nu}{\hat{p}}\begin{pmatrix}0\\ -m\bar{p}\\ -\sqrt{2}p^{+}(p^{-}-\hat{p})\\ 0\end{pmatrix}(p^{2}-m^{2})\,, \tag{4.36}$$

$$(\not{p}+m)v^{-}(p)=\frac{\nu}{\hat{p}}\begin{pmatrix}0\\ -\sqrt{2}p^{+}(p^{-}-\hat{p})\\ mp\\ 0\end{pmatrix}(p^{2}-m^{2})\,. \tag{4.37}$$

Let us compare these solutions with the off-shell definition of the spinor from refs. [37 − 40], which take

$$u(p,\pm)=\frac{\not{p}+m}{\langle p^{\flat}\mp|q\pm\rangle}|q\pm\rangle\,, \tag{4.38}$$

where $|q\pm\rangle$ is a Weyl spinor for an arbitrary null vector q, and $p^{\flat}$ is the null projection of p obtained by subtracting a vector proportional to q. If we choose $|q\pm\rangle$ to be the eigenspinor of helicity $h=\mathbf{p}\cdot\boldsymbol{\sigma}/|\mathbf{p}|$, this spinor has also definite helicity off shell, since h commutes with $\not{p}+m$. Due to the additional term proportional to p^2-m^2 in (4.23) and (4.24), our choices of spinors are not helicity eigenspinors off shell, so our off-shell continuation of the spinors is different from (4.38). Hence our spinors cannot be written in that form. This can also be seen from (4.34)–(4.37): if we multiply (4.38) by $\not{p}-m$, the right hand side is p^2-m^2 multiplied with a Weyl spinor $|q\pm\rangle$, but in (4.34)–(4.37), it is obvious that the right hand side is not a Weyl spinor.

However, we can establish a relation between our spinors and (4.38): in (4.22)–(4.24) the spinors *without* the additional off-shell term have definite helicity, and these can be cast in the form of (4.38). Thus, our choice simply adds an extra term to (4.38):

$$u(p,+)=\frac{\not{p}+m}{\langle p^{\flat}+|q-\rangle}|q-\rangle+\nu\frac{p}{\hat{p}}(p^2-m^2)|\mu,+\rangle, \tag{4.39}$$

$$u(p,-)=\frac{\not{p}+m}{\langle p^{\flat}-|q+\rangle}|q+\rangle+\nu\frac{\bar{p}}{\hat{p}}(p^2-m^2)|\mu,-\rangle, \tag{4.40}$$

where $\mu=(1,0,0,1)/2$ in Minkowski co-ordinates (with spinors $|\mu+\rangle=(0,0,1,0)^{\mathrm{T}}$ and $|\mu-\rangle=(0,-1,0,0)^{\mathrm{T}}$) and $q=(|\mathbf{p}|,-\mathbf{p})$ are null vectors. The specific choice of q here is such that it makes the $\pm$ embellishment of the spinor u coincide with the physical helicity of the fermion; for arbitrary q, this embellishment loses this physical significance.

Let us now use the above equations to make the application of our solutions to the LSZ formalism concrete. For an out-going quark,

$$\begin{aligned}&\bar{u}^{+}(p)\,i(-\not{p}+m)\langle\cdots\psi(p)\cdots\rangle\\ &\qquad=\nu\Big(-\sqrt{2}m\bar{p}\mathcal{V}(\cdots\bar{\alpha}^{-}(-p)\cdots)-2p^{+}(p^{-}-\hat{p})\mathcal{V}(\cdots\bar{\alpha}^{+}(-p)\cdots)\Big),\end{aligned} \tag{4.41}$$

$$\begin{aligned}&\bar{u}^{-}(p)\,i(-\not{p}+m)\langle\cdots\psi(p)\cdots\rangle\\ &\qquad=\nu\Big(\sqrt{2}mp\mathcal{V}(\cdots\bar{\alpha}^{+}(-p)\cdots)-2p^{+}(p^{-}-\hat{p})\mathcal{V}(\cdots\bar{\alpha}^{-}(-p)\cdots)\Big),\end{aligned} \tag{4.42}$$

respectively for positive- and negative-helicity, where $\mathcal{V}$ is the amputated correlation function computed using the light-cone or MHV action. Similarly, for an out-going antiquark,

$$\begin{aligned}&\langle\cdots\bar{\psi}(p)\cdots\rangle i(-\not{p}-m)v^{+}(p)\\&\qquad=\nu\Big(-\sqrt{2}m\bar{p}\mathcal{V}(\cdots\alpha^{-}(-p)\cdots)-2p^{+}(p^{-}-\hat{p})\mathcal{V}(\cdots\alpha^{+}(-p)\cdots)\Big),\end{aligned}\tag{4.43}$$

$$\begin{aligned}&\langle\cdots\bar{\psi}(p)\cdots\rangle i(-\not{p}-m)v^{-}(p)\\&\qquad=\nu\Big(\sqrt{2}mp\mathcal{V}(\cdots\alpha^{+}(-p)\cdots)-2p^{+}(p^{-}-\hat{p})\mathcal{V}(\cdots\alpha^{-}(-p)\cdots)\Big).\end{aligned}\tag{4.44}$$

Note that for definite helicity external quarks, both quark chiralities contribute to the amplitude.

Considering (4.28)–(4.30), we see that the external factors in (4.41)–(4.44) are just the components of the polarisation spinors that correspond to α and $\bar{\alpha}$. Since the additional terms in (4.23) and (4.24) do not affect these components, the external factors are the same as the components of the spinor solutions without the additional off-shell terms. So we conclude that one does not need to consider contributions from the non-dynamical quark components to the amplitudes in light-cone QCD or MHV lagrangian-based calculations. As a check of this decoupling of $\beta^{\pm}$ and $\bar{\beta}^{\pm}$, in section 4.3 we will compare the three point off-shell amplitude $A(1^{\pm}_{\mathrm{q}}2^{-}3^{\pm}_{\bar{\mathrm{q}}})$ calculated using light-cone gauge QCD with the one calculated directly from Feynman gauge QCD. It is worth pointing out that this discussion does not depend on the specific choice of the helicity basis $\eta^{\pm}$ in (4.23) and (4.24); one can also choose a different q for (4.39) and (4.40) and a different coefficient before the factor p^2-m^2, repeat the foregoing, and arrive at the same conclusion.

4.2 Helicity flipping property of the amplitude

We can now look at what happens to the amplitude when we flip the external helicities. For amplitudes of massless QCD, we know that under such an operation, the corresponding amplitude exchanges the holomorphic and anti-holomorphic variables. We will show that in the amplitude with massive quarks, this property should be modified a little: we need to multiply the result by $(-1)^f$ where $f=\frac{1}{2}(n_+-n_-)$. Here, $n_\pm$ is the total number of positive (negative) helicity quarks and antiquarks.

For a quark-antiquark pair, there could be four kinds of terms in the amplitude: terms composed of the amputated correlation functions with $\bar{\alpha}^+\alpha^+$, $\bar{\alpha}^+\alpha^-$, $\bar{\alpha}^-\alpha^+$, $\bar{\alpha}^-\alpha^-$ and corresponding coefficients from LSZ formulae (4.41)–(4.44). From the first two equations, we can see when the helicity of the quark is flipped from $+$ to $-$, the factors corresponding to $\bar{\alpha}^-$ changes to the one corresponding to $\bar{\alpha}^+$ by exchanging the the holomorphic variable and anti-holomorphic variables and changing the overall sign, i.e. $-\sqrt{2}m\bar{p}\to\sqrt{2}mp$, whereas the term with $\bar{\alpha}^+\to\bar{\alpha}^-$ exchanges the holomorphic and anti-holomorphic variables without changing the sign. And when the helicity changes from $-$ to $+$, the factors behave in just the opposite fashion. For the antiquark, there is a similar property. This has been summarised in table 3.

As a result, the flipped amplitude can be obtained from the original amplitude by the following operations: changing chirality of corresponding $\bar{\alpha}\alpha$ pair and helicity of A

	$\bar\alpha^+ \to \bar\alpha^-$	$\bar\alpha^- \to \bar\alpha^+$
$q^+ \to q^-$	$+$	$-$
$q^- \to q^+$	$-$	$+$

	$\alpha^+ \to \alpha^-$	$\alpha^- \to \alpha^+$
$\bar q^+ \to \bar q^-$	$+$	$-$
$\bar q^- \to \bar q^+$	$-$	$+$

Table 3: The sign changes that follow from the LSZ formulae of (4.41)–(4.44).

	$\bar\alpha^+\alpha^+ \leftrightarrow \bar\alpha^-\alpha^-$	$\bar\alpha^+\alpha^- \leftrightarrow \bar\alpha^-\alpha^+$
$q^+\bar q^+ \leftrightarrow q^-\bar q^-$	$+$	$-$
$q^-\bar q^+ \leftrightarrow q^+\bar q^-$	$-$	$+$

Table 4: The corresponding sign changes for quark-antiquark pairs that follow from table 3.

of the amputated correlation function, exchanging the holomorphic and anti-holomorphic variables of corresponding external quark factor and at the same time making the sign changes in corresponding terms as table 4 for each quark-antiquark pair.

Moreover, the two chirality violating terms in lagrangian (4.3) and (4.4) swap when we exchange the holomorphic and anti-holomorphic variables, flip the chirality and change the overall sign, while the other chirality conserving vertices in lagrangian from (2.22) to (2.26) do not change sign under this operation. This means that for an external like-chirality $\alpha^\pm\bar\alpha^\pm$ pair connected by an internal quark line, the operation corresponding to the first column in the above table contributes a minus sign to the amputated correlation function after we exchange the holomorphic and anti-holomorphic variables, because there should be an odd number of chirality violating vertices on the fermion line connecting these two fermions. Multiplying the first column with this sign change, we arrive at the sign changes we need to apply for each quark-antiquark pair, after exchanging all holomorphic and anti-holomorphic variables. So we can see that for a like-helicity quark-antiquark pair there will be a minus sign. It is clear that there is an even (odd) number of like-helicity quark antiquark pairs when f is even (odd), and thus our original assertion is proved.

4.3 The three point amplitude $A(1^\pm_{\rm q}2^-3^\pm_{\bar{\rm q}})$

In this section we will compare the amplitude $A(1^\pm_{\rm q}2^-3^\pm_{\bar{\rm q}})$ calculated using the light cone lagrangian combined with LSZ formulae (4.41)–(4.44), with the one computed using Feynman gauge QCD using the same $\bar u(p)$ and $v(p)$ from (4.28)–(4.30). As we noted earlier, this is in order to check the conclusion in section 4.1 that one does not need to consider the contribution of the non-dynamical quark variables to the amplitudes when using light-cone QCD and MHV lagrangian methods. Notice here that the notation $A(1^\pm_{\rm q}2^-3^\pm_{\bar{\rm q}})$ denotes *four* amplitudes (not two), i.e. the quark helicity superscripts are independent.

In Feynman gauge QCD the only interaction term which contributes to the amplitude is $\frac{g}{\sqrt2}\bar\psi \not{A}\psi$ and the amplitude can be read out directly:

$$A(1^\pm_{\rm q}2^-3^\pm_{\bar{\rm q}}) = -i\frac{g}{\sqrt2}\bar u_1^\pm \not{A}_2^- v_3^\pm, \tag{4.45}$$

the minus sign coming from the fermion statistics. Let us look at $A(1^+_{\rm q}2^-3^-_{\bar{\rm q}})$ first: by using the explicit expression of $\not{A}^-$ with $\mu = (1,0,0,1)/\sqrt2$ as the reference momentum in the gluon's polarisation vector,

$$\not{A}^- = \sqrt{2}\begin{pmatrix} & & 0 & 0 \\ & & -1 & -k/\hat{k} \\ -k/\hat{k} & 0 & & \\ 1 & 0 & & \end{pmatrix}, \tag{4.46}$$

(where the missing entries are all 0, omitted for clarity) and equation (4.28), we eventually obtain:

$$\begin{aligned} A(1_{\rm q}^+ 2^- 3_{\bar{\rm q}}^-) = -ig\nu_1\nu_3\Big[2m^2\Big(\frac{\bar{1}\tilde{2}\tilde{3}}{\hat{2}} - \frac{\tilde{3}}{\hat{3}}(1^- - \hat{1})(\hat{3}+1^+)\Big) \\ +4\frac{1^+3^+(2\,3)}{\hat{2}\hat{3}}(1^- - \hat{1})(3^- - \hat{3})\Big]. \end{aligned} \tag{4.47}$$

Notice that we have only used momentum conservation here, but not the on-shell condition in the calculation, since the three external particles cannot be all on shell at the same time. One can easily check that when $m \to 0$, this amplitude recovers the massless off-shell amplitude.

Let us repeat this exercise using massive light-cone QCD, the terms of which that contribute to the amplitudes are (2.24) and (4.3). Reformulating them in momentum space now converting to the canonical normalisation (i.e. such that the action is normalised as $S = \int d^4x \sum \tilde{L}$, *cf.* (2.17)) for convenience:

$$\tilde{L}^{\bar{\psi}-\psi} = g\Big[\frac{(1\,2)}{\hat{1}\hat{2}}\bar{\alpha}_{\bar{1}}^- \bar{A}_{\bar{2}}\alpha_{\bar{3}}^+ - \frac{(2\,3)}{\hat{2}\hat{3}}\bar{\alpha}_{\bar{1}}^+ \bar{A}_{\bar{2}}\alpha_{\bar{3}}^-\Big], \tag{4.48}$$

$$\tilde{L}^{\bar{\psi}^+-\psi^+} = \frac{gm}{\sqrt{2}}\frac{\hat{2}}{\hat{1}\hat{3}}\bar{\alpha}_{\bar{1}}^+ \bar{A}_{\bar{2}}\alpha_{\bar{3}}^+ . \tag{4.49}$$

After extracting the vertices from the above two equations and using equations (4.41) and (4.44) we can write down:

$$\begin{aligned} A(1_{\rm q}^+ 2^- 3_{\bar{\rm q}}^-) = -i\nu_1\nu_3\bar{E}_2\Big[\sqrt{2}m\bar{1} \times \Big(g\frac{(1\,2)}{\hat{1}\hat{2}}\Big) \times (-\sqrt{2}m\tilde{3}) \\ +4 \times 1^+(\hat{1} - 1^-) \times \Big(-g\frac{(2\,3)}{\hat{2}\hat{3}}\Big) \times (\hat{3} - 3^-)3^+ \\ +2 \times 1^+(\hat{1} - 1^-) \times \Big(\frac{gm}{\sqrt{2}}\frac{\hat{2}}{\hat{1}\hat{3}}\Big) \times (-\sqrt{2}m\tilde{3})\Big], \end{aligned} \tag{4.50}$$

where $\bar{E}_2 = -1$ is the polarization of $\bar{A}_2$. The second term above is just the second term of (4.47), and using the fact that $p\bar{p} = -(\hat{p} - p^+)(\hat{p} - p^-)$, we see that the first and third terms above sum to the first term in (4.47). This validates (at least in this particular case) the use of the Feynman rules derived from the light-cone QCD action, when combined with the LSZ reduction as in (4.41)–(4.44). Furthermore, we note that this also validates the use of the Feynman rules obtained from MHV lagrangian as well, since (4.48) and (4.49) both correspond to terms found therein by leading-order substitutions into (2.24) and (4.3), respectively.

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Similarly, we can also calculate the other three amplitudes:

$$
\begin{aligned}
A(1_{\rm q}^- 2^- 3_{\bar{\rm q}}^+) = -ig\nu_1\nu_3 \Big[2m^2 \Big(\frac{\tilde{1}\tilde{2}\bar{3}}{\hat{2}} - \frac{\tilde{1}}{\hat{1}}(3^- - \hat{3})(\hat{1} + 3^+) \Big) \\
- 4\frac{1^+ 3^+ (1\,2)}{\hat{1}\hat{2}}(\hat{1} - 1^-)(\hat{3} - 3^-) \Big] ,
\end{aligned} \tag{4.51}
$$

$$
\begin{aligned}
A(1_{\rm q}^+ 2^- 3_{\bar{\rm q}}^+) = -ig\nu_1\nu_3 \Big[2\sqrt{2}m \Big(-\frac{\bar{1}\tilde{2}3^+}{\hat{2}}(3^- - \hat{3}) - \frac{1^+\tilde{2}\bar{3}}{\hat{2}}(1^- - \hat{1}) \\
+ (1^- - \hat{1})(3^- - \hat{3})(1^+ + 3^+) \Big) \Big] ,
\end{aligned} \tag{4.52}
$$

$$
\begin{aligned}
A(1_{\rm q}^- 2^- 3_{\bar{\rm q}}^-) = -ig\nu_1\nu_3 \Big[2\sqrt{2}m \Big(\frac{1^+\tilde{3}\,(1\,2)}{\hat{1}\hat{2}}(1^- - \hat{1}) - \frac{\tilde{1}3^+(2\,3)}{\hat{2}\hat{3}}(3^- - \hat{3}) \Big) \\
- \sqrt{2}m^3 \frac{\tilde{1}\hat{2}\tilde{3}}{\hat{1}\hat{3}} \Big] .
\end{aligned} \tag{4.53}
$$

We can see that $A(1_{\rm q}^+ 2^- 3_{\bar{\rm q}}^+)$ and $A(1_{\rm q}^- 2^- 3_{\bar{\rm q}}^-)$ are symmetric under $1 \leftrightarrow 3$ and vanish in the massless limit.

4.4 The three point amplitude: $A(1_{\rm q}^\pm 2^+ 3_{\bar{\rm q}}^\pm)$

In this section, we will compare the amplitudes $A(1_{\rm q}^\pm 2^+ 3_{\bar{\rm q}}^\pm)$ calculated using light-cone gauge massive QCD with those from the massive MHV lagrangian in order to check that our massive MHV lagrangian scenario is valid and equivalent to ordinary perturbation method.

In light-cone QCD, we would expect contributions to this amplitude from (2.23) and (4.4). On the other hand, in the MHV lagrangian (2.23) has been eliminated by the transformation, but the amplitude will receive contributions from the leading-order substitutions for the new fields into (4.4), and the next-to-leading-order terms of the transformation into (4.2). We should still keep in mind that there are MHV completion vertex contributions which should be taken into account since we are dealing with the off-shell amplitude. Therefore the sum of the contributions from both (4.2) and completion vertices on the MHV lagrangian side should be equal to the one from (2.23) in light-cone QCD.

First, consider the $\mathcal{O}(B)$ terms in the expansion of (4.2) in terms of the new fields (again, changing to the canonical normalisation):

$$
\tilde{L}_m^{\bar{\psi}\psi} = \frac{m^2 g}{2} \left(\frac{\hat{2}}{(1\,2)\hat{3}} \bar{\xi}_{\bar{1}}^- B_{\bar{2}} \xi_{\bar{3}}^+ - \frac{\hat{2}}{(1\,2)\hat{1}} \bar{\xi}_{\bar{1}}^+ B_{\bar{2}} \xi_{\bar{3}}^- \right) . \tag{4.54}
$$

As an example, let us look at $A(1_{\rm q}^+ 2^+ 3_{\bar{\rm q}}^-)$. From the LSZ reduction formula (4.41) and (4.44), we can write down the contribution to the amplitude from the vertex in the above term:

$$
\begin{aligned}
i\frac{m^2 g}{2}\nu_1\nu_3 \Big[(\sqrt{2}m\bar{1}) \times \left(\frac{\hat{2}}{(1\,2)\hat{3}} \right) \times (-\sqrt{2}m\tilde{3}) \\
+4 \times 1^+(1^- - \hat{1}) \times \left(-\frac{\hat{2}}{(1\,2)\hat{1}} \right) \times (3^- - \hat{3})3^+ \Big] .
\end{aligned} \tag{4.55}
$$

Next we calculate the MHV completion vertices' contribution to the correlation functions $\langle\alpha_1^+\bar{A}_2\bar{\alpha}_3^-\rangle$ and $\langle\alpha_1^-\bar{A}_2\bar{\alpha}_3^+\rangle$ and use the LSZ formula. We expand the fields to next-to-leading order, taking the $B\xi$ term for α_1, the $\xi\bar{\xi}$ term for $\bar{B}_2$, and the $\bar{\xi}B$ term for $\bar{\alpha}_3$ to obtain:

$$\begin{aligned}
\left\langle \alpha_{1}^{+}{}_{i}\,\bar{A}_2^{a_2}\bar{\alpha}_3^{-\bar{\jmath}}\right\rangle &= -\frac{g}{\sqrt{2}}\frac{\hat{2}+\hat{3}}{(2\,3)}\left\langle \left(B_{\bar{2}}\xi_{\bar{3}}^{+}\right)_i\bar{B}_2^{a_2}\bar{\xi}_3^{-\bar{\jmath}}\right\rangle - \frac{g}{\sqrt{2}}\frac{\hat{1}}{(1\,2)}\left\langle \xi_1^{+}{}_i\,\bar{B}_2^{a_2}\left(\bar{\xi}_{\bar{1}}^{-}B_{\bar{2}}\right)^{\bar{\jmath}}\right\rangle \\
&\quad -\frac{g}{2}\frac{\hat{1}}{\hat{2}(1\,3)}\left\langle \xi_1^{+}{}_i\left(\bar{\xi}_{\bar{1}}^{-}T^{a_2}\xi_{\bar{3}}^{+}\right)\bar{\xi}_3^{-\bar{\jmath}}\right\rangle \\
&= \frac{ig}{2}\frac{i\sqrt{2}\,\hat{1}}{(p_1^2-m^2)}\frac{i}{p_2^2}\frac{i\sqrt{2}\,\hat{3}}{p_3^2-m^2}\left(2\times\frac{\{2\,3\}}{\hat{2}\hat{3}}+m^2\frac{\hat{2}}{(1\,2)\hat{3}}\right)(T^{a_2})_i{}^{\bar{\jmath}}, \qquad (4.56)\\
\left\langle \alpha_{1}^{-}{}_{i}\,\bar{A}_2^{a_2}\bar{\alpha}_3^{+\bar{\jmath}}\right\rangle &= -\frac{g}{\sqrt{2}}\frac{\hat{3}}{(2\,3)}\left\langle \left(B_{\bar{2}}\xi_{\bar{3}}^{-}\right)_i\bar{B}_2^{a_2}\bar{\xi}_3^{+\bar{\jmath}}\right\rangle - \frac{g}{\sqrt{2}}\frac{\hat{1}+\hat{2}}{(1\,2)}\left\langle \xi_1^{-}{}_i\,\bar{B}_2^{a_2}\left(\bar{\xi}_{\bar{1}}^{+}B_{\bar{2}}\right)^{\bar{\jmath}}\right\rangle \\
&\quad +\frac{g}{2}\frac{\hat{3}}{\hat{2}(1\,3)}\left\langle \xi_1^{-}{}_i\left(\bar{\xi}_{\bar{1}}^{+}T^{a_2}\xi_{\bar{3}}^{-}\right)\bar{\xi}_3^{+\bar{\jmath}}\right\rangle \\
&= -\frac{ig}{2}\frac{i\sqrt{2}\,\hat{1}}{(p_1^2-m^2)}\frac{i}{p_2^2}\frac{i\sqrt{2}\,\hat{3}}{p_3^2-m^2}\left(2\times\frac{\{1\,2\}}{\hat{1}\hat{2}}+m^2\frac{\hat{2}}{(1\,2)\hat{1}}\right)(T^{a_2})_i{}^{\bar{\jmath}}. \qquad (4.57)
\end{aligned}$$

By writing it this way, with the propagators made manifest, it is easy to read off the amputated correlation function from these equations. Notice that the second term in the big bracket of (4.56) is just the same as the coefficient of the first term in the parentheses of (4.54). As a result, the first term in (4.55) is cancelled with the contribution from second term in (4.56) after LSZ procedure. Similarly, the second term in (4.55) is cancelled with the contribution from the second term in (4.57). So the only terms left are the first terms of (4.56) and (4.57). It is worth noticing that this cancellation happens before we apply the LSZ procedure. We conclude that this cancellation holds for all four amplitudes $A(1_{\mathrm{q}}^{\pm}2^{+}3_{\bar{\mathrm{q}}}^{\pm})$. Note that it is impossible to send all three external particles on shell at the same time in Minkowski signature, and these completion vertices contribute to the off-shell amplitude. If one goes to $(2,2)$ signature, one *can* discuss the on-shell amplitude, and in this case, one can easily verify that these completion vertices give zero contribution using the on-shell identity

$$(1\,2)\{1\,2\} = (2\,3)\{2\,3\} = (1\,3)\{1\,3\} = -\frac{m^2}{2}\hat{2}^2. \qquad (4.58)$$

At tree level, the completion vertices make no contribution to the higher point amplitudes. This is clear because these vertices can survive LSZ reduction only by diverging, but by inspection they generically diverge only for collinear momenta. Since in the higher point vertices the kinematical constraints are insufficient to force the on-shell momenta to be collinear, none of these terms survive the on-shell limit.

Next, we turn to (2.23) from the light-cone QCD lagrangian, which in momentum space is

$$L^{\bar{\psi}+\psi} = g\left(\frac{\{1\,2\}}{\hat{1}\hat{2}}\bar{\alpha}_{\bar{1}}^{+}A_{\bar{2}}\alpha_{\bar{3}}^{-} - \frac{\{2\,3\}}{\hat{2}\hat{3}}\bar{\alpha}_{\bar{1}}^{-}A_{\bar{2}}\alpha_{\bar{3}}^{+}\right). \qquad (4.59)$$

It is clear that what this contributes to the amplitude is just what is left in (4.56) and (4.57) (i.e. their first terms), which we just calculated using the MHV lagrangian. (The relative minus sign in (4.56) and (4.57) comes from fermion statistics.) From the foregoing it is clear that this holds for all four amplitudes.

As a completion, we provide the four amplitudes $A(1_{\mathrm{q}}^{\pm}2^{+}3_{\bar{\mathrm{q}}}^{\pm})$:

$$\begin{aligned} A(1_{\mathrm{q}}^{+}2^{+}3_{\bar{\mathrm{q}}}^{-}) = -ig\nu_1\nu_3\Big[2m^2\Big(\frac{\bar{1}\bar{2}\tilde{3}}{\hat{2}} - \frac{\bar{1}}{\hat{1}}(3^{-}-\hat{3})(\hat{1}+3^{+})\Big) \\ - 4\frac{1^{+}3^{+}\{1\,2\}}{\hat{1}\hat{2}}(1^{-}-\hat{1})(3^{-}-\hat{3})\Big], \end{aligned} \tag{4.60}$$

$$\begin{aligned} A(1_{\mathrm{q}}^{-}2^{+}3_{\bar{\mathrm{q}}}^{+}) = -ig\nu_1\nu_3\Big[2m^2\Big(\frac{\tilde{1}\bar{2}\bar{3}}{\hat{2}} - \frac{\bar{3}}{\hat{3}}(1^{-}-\hat{1})(\hat{3}+1^{+})\Big) \\ + 4\frac{1^{+}3^{+}\{2\,3\}}{\hat{2}\hat{3}}(1^{-}-\hat{1})(3^{-}-\hat{3})\Big], \end{aligned} \tag{4.61}$$

$$\begin{aligned} A(1_{\mathrm{q}}^{-}2^{+}3_{\bar{\mathrm{q}}}^{-}) = -ig\nu_1\nu_3\Big[2\sqrt{2}m\Big(\frac{1^{+}\bar{2}\tilde{3}}{\hat{2}}(1^{-}-\hat{1}) + \frac{\tilde{1}\bar{2}3^{+}}{\hat{2}}(3^{-}-\hat{3}) \\ - (1^{-}-\hat{1})(3^{-}-\hat{3})(1^{+}+3^{+})\Big)\Big], \end{aligned} \tag{4.62}$$

$$\begin{aligned} A(1_{\mathrm{q}}^{+}2^{+}3_{\bar{\mathrm{q}}}^{+}) = -ig\nu_1\nu_3\Big[2\sqrt{2}m\Big(\frac{\bar{1}3^{+}\{2\,3\}}{\hat{2}\hat{3}}(3^{-}-\hat{3}) - \frac{1^{+}\bar{3}\{1\,2\}}{\hat{1}\hat{2}}(1^{-}-\hat{1})\Big) \\ + \sqrt{2}m^3\frac{\bar{1}\hat{2}\bar{3}}{\hat{1}\hat{3}}\Big]. \end{aligned} \tag{4.63}$$

Comparing with the results in the previous section one can also see that flipping all external helicities corresponds to changing the holomorphic variables to anti-holomorphic variables in the first two amplitudes, together with an extra minus sign in the last two amplitudes as we proved before.

5. Conclusion

In this paper, we extended the canonical MHV lagrangian formalism of [22, 24, 25], and thus proved that we can obtain a canonical MHV lagrangian formalism for massless QCD theory with $\mathrm{SU}(N_{\mathrm{C}})$ gauge symmetry. We started with massless QCD in the light-cone gauge with the non-dynamical field components integrated out. By applying a canonical transformation to the field variables, we obtain a lagrangian incorporating gluon-gluon and quark-gluon interactions whose vertices must be proportional (up to polarisation factors) to the MHV amplitudes in the literature. For completeness, this has been checked explicitly for amplitudes with two quarks and two gluons, with four quarks, and with two quarks and three gluons in the $(1_{\mathrm{q}}^{+}2^{+}3^{-}4^{+}5_{\bar{\mathrm{q}}}^{-})$ configuration.

The MHV QCD lagrangian we have found maintains 'backward compatibility' with the pure-gauge case found in [22, 24]. We note that the pure-gauge part is preserved, as is the solution for $\mathcal{A}$ in terms of $\mathcal{B}$. In contrast $\bar{\mathcal{A}}$ acquires new terms in the new fermion fields brought on by the requirement that the transformation is canonical. As in the pure-gauge

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case, the explicit form of this transformation as a series expansion has coefficients that have simple, holomorphic expressions in the momenta. We summarise these results below:

$$
\begin{aligned}
\mathcal{A}_1 &= \sum_{n=2}^{\infty} \int_{2\cdots n} \Upsilon(1\cdots n)\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\bar{n}}\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^{n}\mathbf{p}_i)\\
\bar{\mathcal{A}}_1 &= \sum_{m=2}^{\infty}\sum_{s=2}^{m} \int_{2\cdots m} \frac{\hat{s}^2}{\hat{1}^2}\Upsilon(1\cdots m)\mathcal{B}_{\bar{2}}\cdots\bar{\mathcal{B}}_{\bar{s}}\cdots\mathcal{B}_{\bar{m}}\,(2\pi)^3\delta(\textstyle\sum_{i=1}^{m}\mathbf{p}_i)\\
&\quad + \frac{g^2}{\hat{1}^2\sqrt{8}}\sum_{\pm}\sum_{n=3}^{\infty}\sum_{j=1}^{n-2}\int_{2\cdots n}\begin{Bmatrix}-\widehat{j+1}\\ \widehat{j+2}\end{Bmatrix}\Upsilon(1\cdots n)\\
&\quad \times\left\{\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\bar{j}}\xi^{\mp}_{\overline{j+1}}\bar{\xi}^{\pm}_{\overline{j+2}}\mathcal{B}_{\overline{j+3}}\cdots\mathcal{B}_{\bar{n}} + \frac{1}{N_{\mathrm{C}}}\bar{\xi}^{\pm}_{\overline{j+2}}\mathcal{B}_{\overline{j+3}}\cdots\mathcal{B}_{\bar{n}}\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\bar{j}}\xi^{\mp}_{\overline{j+1}}\right\}\\
&\quad \times(2\pi)^3\delta^3(\textstyle\sum_{i=1}^{n}\mathbf{p}_i),\\
\bar{\alpha}_1^{\pm} &= \bar{\xi}_1^{\pm} + \sum_{n=3}^{\infty}\int_{2\cdots n}\begin{Bmatrix}1\\ -\hat{1}/\hat{2}\end{Bmatrix}\Upsilon(1\cdots n)\,\bar{\xi}^{\pm}_{\bar{2}}\mathcal{B}_{\bar{3}}\cdots\mathcal{B}_{\bar{n}}\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^{n}\mathbf{p}_i),\\
\alpha_1^{\pm} &= \xi_1^{\pm} + \sum_{n=3}^{\infty}\int_{2\cdots n}\begin{Bmatrix}1\\ -\hat{n}/\hat{1}\end{Bmatrix}\Upsilon(1\cdots n)\,\mathcal{B}_{\bar{2}}\cdots\mathcal{B}_{\overline{n-1}}\xi^{\pm}_{\bar{n}}\,(2\pi)^3\delta^3(\textstyle\sum_{i=1}^{n}\mathbf{p}_i),
\end{aligned}
$$

where the stacked expressions in braces take their value in accordance with the upper or lower choice of sign, and

$$
\Upsilon(1\cdots n) = (-i)^n\frac{\hat{1}\hat{3}\cdots\widehat{n-1}}{(2\,3)\cdots(n-1,n)}.
$$

We have also begun the analysis of the MHV lagrangian for QCD with massive quarks. We have observed that we can retain much of the structure from the massless case, and although we defer a detailed analysis of the amplitudes following from this lagrangian, we see that the process results again in an infinite series of terms, but where the chirality rather than the helicity content of each term is constrained. In the process, we derived explicitly the closed-form expressions for the new vertices that arise in the massive MHV lagrangian which are proportional to m and m^2. We cannot comment here whether or not such an approach would yield a good computational alternative to established techniques such as BCFW recursion relations for massive particles [35, 36, 40 – 44], but we remark that the important question is whether it helps simplify the calculation of amplitudes involving massive quarks at next-to-leading order and beyond.

As we found previously in [25], completion vertices, arising from the terms in the transformation itself, are necessary to recover (parts of) otherwise missing amplitudes. We have verified this in a simple case for massless QCD. They are of course necessary for recovering the full off-shell theory [25], which we showed in some example cases for massive QCD. We expect these completion vertices to be important at the loop level.

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A. Proofs of field transformation coefficients

In this appendix, we provide a proof for the expression (2.53) for S^- and outline the proof of (2.67) for K^-, as given in section 2.5.

A.1 S^- coefficients

S^- has coefficients defined by the recurrence relations (2.51). Using (2.53) to substitute for S^-, the r.h.s. of (2.51) becomes

$$\begin{aligned}(-i)^n \frac{\hat{1}\hat{4}\cdots\widehat{n-1}}{(3\,4)\cdots(n{-}1,n)} &\left\{\frac{\hat{3}}{(3,2{+}P_{3n})} + \sum_{j=3}^{n-1} \frac{(j,j{+}1)}{(j,2{+}P_{j,n})}\frac{\hat{2}+\hat{P}_{j+1,n}}{(j{+}1,2{+}P_{j+1,n})}\right\} \\ = (-i)^n \frac{\hat{1}\hat{4}\cdots\widehat{n-1}}{(3\,4)\cdots(n{-}1,n)} &\left(x_3 + \sum_{j=3}^{n-1} y_j\right)\end{aligned} \tag{A.1}$$

where

$$y_j = \frac{(j,j{+}1)}{(j,2{+}P_{jn})}\frac{\hat{2}+\hat{P}_{j+1,n}}{(j{+}1,2{+}P_{j+1,n})} \quad \text{and} \quad x_j = \frac{\hat{\jmath}}{(j,2{+}P_{jn})}. \tag{A.2}$$

Now notice that $x_j + y_j = x_{j+1}$, so the sum on the r.h.s. of (2.51) collapses to

$$(-i)^n \frac{\hat{1}\hat{4}\cdots\widehat{n-1}}{(3\,4)\cdots(n{-}1,n)} x_n = S^-(12;3\cdots n), \tag{A.3}$$

and the proof is complete.

A.2 K^- coefficients

Here we will outline the proof that expression (2.67) for K^- satisfies the recurrence relation (2.60). As noted in the main text, the proof is by induction on n. We treat the $n = 3$ case separately since it involves only the last term on the r.h.s. of (2.60).

For higher n, we substitute (2.67) into the recurrence relation. For each term on the r.h.s. of (2.60), we can pull out a factor of

$$\frac{\hat{1}\cdots\hat{n}}{(1\,2)\cdots(n\,1)}$$

leaving telescoping sums of the form

$$\sum_{j=a}^{b} \hat{P}_{ij}\left(\frac{\widetilde{j+1}}{\widehat{j+1}} - \frac{\tilde{\jmath}}{\hat{\jmath}}\right) = \hat{P}_{ib}\frac{\widetilde{b+1}}{\widehat{b+1}} - \hat{P}_{ia}\frac{\tilde{a}}{\hat{a}} - \tilde{P}_{a+1,b}. \tag{A.4}$$

(In the second term of (2.60), two such sums are nested.) The cases noted below (2.60), where a sum is taken to vanish if its upper limit is less than its lower limit, can be handled consistently by defining the sum $P_{ij} = p_i + p_{i+1} + \cdots + p_n + p_1 + \cdots + p_j$ when $j < i$. When this is so and we evaluate the sums using (A.4), we find that they vanish in these particular conditions because of terms of the form $P_{i,i-1} = 0$.

To complete the proof, one simply evaluates the sums and does the algebra while applying the conservation of momentum. This results in an expression on the r.h.s. of (2.60) equal to the given for $K^{-(j)}(1 \cdots n)$ in (2.67).

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